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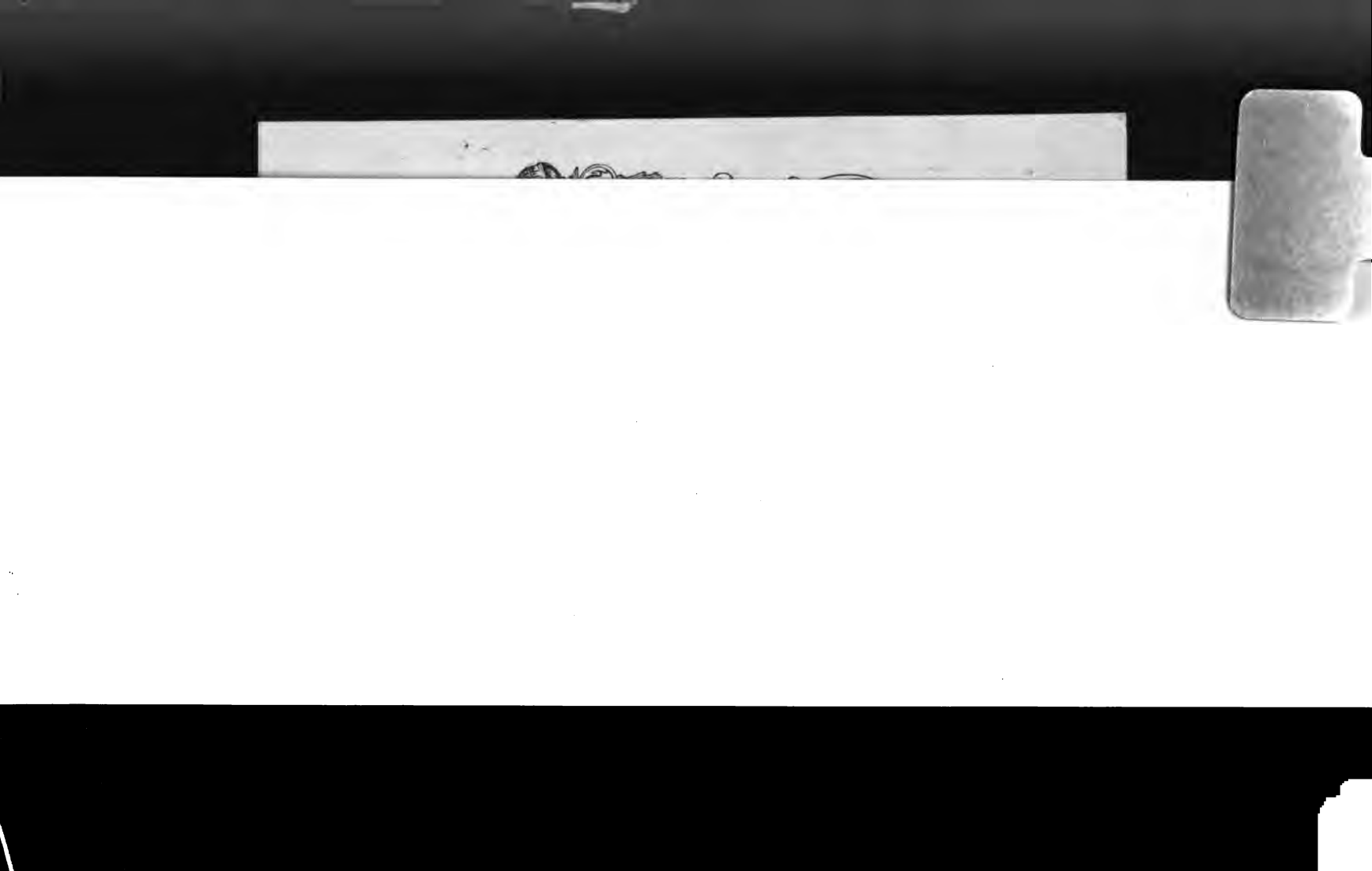
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DISCUSSION

OF THE

PRECISION OF MEASUREMENTS.

WITH EXAMPLES TAKEN MAINLY FROM

PHYSICS AND ELECTRICAL ENGINEERING.

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PREFACE.

THE material presented in this volume is the outcome of several years' teaching of the subject. In a less complete form it was prepared for lecture notes and was printed in pamphlet form, but not published, by the Massachusetts Institute of Technology in 1888, having appeared in the *Technology Quarterly* and in the *Electrical Engineer* in 1887.

In this revised form, the author has felt that it perhaps possessed sufficient completeness and originality to be of interest or value to students and teachers, and therefore to merit publication.

In venturing to urge the importance of the subject as a course of study for engineers and for students of physics or other pure sciences, the author would suggest the value of the attitude of mind produced by it. One who has in any reasonable degree mastered its methods, although he may never apply them directly, will not only have increased his power to intelligently scrutinize experimental results, but will have acquired a tendency to do so. And it is perhaps not too much to hope that he may acquire a notion of a judicious distribution of effort which, with the best of results to himself, he may carry into quite other matters.

SILAS W. HOLMAN.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, September, 1892.

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PRECISION OF MEASUREMENTS.

INTRODUCTORY.

AN experimental result whose reliability is unknown is nearly worthless. The grade of accuracy of a measurement must be adapted to the purpose for which the result is desired. The necessary accuracy must be secured with the least possible expenditure of labor.

These statements apply no less to the roughest than to the most elaborate work which the engineer is called upon to perform; they are no more true of refined scientific research than of every-day engineering and industrial practice. The principles which underlie these assertions respecting quantitative measurement differ in no essential particular from those which lie at the foundation of all commercial and industrial economy, —proved value; product, labor, and expenditure proportioned to the relative importance of the thing in hand; results obtained with the least effort, and hence with judicious distribution of effort among the various parts of the work.

The successful business manager does not hesitate at large expenditure of money or effort in those parts of an undertaking where he perceives them to be necessary, nor does he overlook the importance of economy where expenditure is not essential. Neither does he wait till an enterprise is well under way or completed to determine where the chief points for expenditure or economy lie. The wise designer of a structure

devotes his close attention to distributing material to the best advantage: enough, at the best points, and no superfluity. These things are so obvious, and their neglect is so strikingly absurd, that it is the more surprising that the same practice should be so commonly neglected not only in quantitative measurement but in engineering investigations and even in physical research.

The engineer, in the measurement of the efficiency or duty of an engine, the efficiency of a dynamo or of a power station; the physicist in the designing or use of a gas thermometer, in the measurement of an index of refraction, or in the comparisons of standards of length; the chemist in analytical investigation, or in the experimental test of an industrial plant,—can no more afford to omit a preliminary discussion of the precision of the various component measurements entering into his result, than the business man can afford to estimate and proportion in advance his expenditure in a large undertaking. The one is as essential as the other to complete success.

The thoughtful student recognizes early in his experimental work the importance of certain questions which never leave the mind of the experienced observer, namely—What accuracy is desired in the result? What accuracy is therefore necessary in each of the various component measurements from which the result is calculated? How reliable is the final result when obtained? The more complicated and indirect the measurement, the more difficult it becomes to answer these queries by mere inspection, and hence the greater the necessity for some systematic and rational procedure for reaching the answer. The present volume is the outcome of an effort to establish such a procedure which, while being sufficiently general, shall not be too laborious in its operation. It is intended to be applicable to quantitative measurements of all kinds, whether in engineering or pure science. The illustrative examples throughout the text are taken chiefly from physics and electrical engineering, because the students as well as the problems with which the author has been called upon to deal have been chiefly in those subjects. The examples are for the most part

so fully explained or so simple that they will be easily intelligible to students in other lines. The processes of the differential calculus have been used, because without them the methods would necessarily be cumbrous, and also because a large and increasing proportion of those who deal at all with such a subject are amply competent to follow or make such simple differentiations as are required. It is, however, to be noted that the majority of the methods and formulæ herein developed can be utilized without any employment whatever of the calculus, so that they may be applied by one who has forgotten his earlier knowledge of that subject or who has never become acquainted with it. Attention is particularly directed in this connection to the rules for significant figures.

DIRECT MEASUREMENTS.

Direct Measurements.—All quantitative work of course involves measurements. These may be separated into two classes, viz. direct and indirect. Direct measurements are those made by methods and instruments whose indications give directly the quantity sought; e.g. measurements of distance by a scale, of weights (or masses) by an equal-arm balance, of resistance by a Wheatstone bridge, etc. The direct readings, in such cases, may or may not require corrections. The fact that a correction is necessary, that is, that the directly observed value must be more or less modified to remove the effect of known sources of error, does not render the measurement indirect.

Indirect Measurements are those in which the quantity measured is not given directly by observation or readings taken, but must be calculated from them. Thus in an indirect measurement the quantity sought is a function of one or more quantities which are directly measured and which may be called the component quantities. For instance, the specific gravity of a substance is ordinarily found by measuring its weight in air and its loss of weight in water. Each of these is or may be a direct measurement, but the desired specific gravity is found from them by calculation, viz. by dividing one by the other, and is therefore indirect. To measure a

constant current C, we may pass it through a tangent galvanometer of factor k and observe the deflection ϕ produced: then $C = k \tan \phi$. Here C is indirectly measured, being calculated from the directly observed deflection ϕ by the function indicated by the right-hand member of the expression. Similarly the measurement of g by a simple pendulum, of the E. M. F. of a battery by the two-deflection method or by the Poggendorff method, of the index of refraction of a prism, of the efficiency of a dynamo or motor,—in fact the great majority of physical measurements,—are indirect.

Quantities may be either ***independent*** or ***conditioned.*** That is, two or any number of quantities to be measured may be wholly independent, so that the magnitude of one is in no way predetermined by any relations to the others; or they may be conditioned so that, for instance, the magnitude of two out of three being given, that of the third is thereby predetermined. Thus a constant current flowing through a given coil of wire might be anything whatever, according to the potential used, so that measurements of current and of resistance at any instant would be independent. But if the potential difference at the ends of the coil were measured simultaneously with the current and resistance, then the three, current, resistance, and potential, would be conditioned by Ohm's law. The numerical values obtained in the measurement of conditioned quantities contain, of course, errors not controlled by these conditions, so that these values fail to fulfil the conditions, and require adjustment.

Sources of Error in Direct Measurements.—All processes of measurement are, of course, fallible. None can give absolute accuracy, that is, none can be wholly free from error. The questions with which we have to deal then are only such as relate to the amount or character of the errors occurring, and to their *sufficient* elimination for the purpose in hand.

Inspection of the methods, instruments, and results of any direct measurement will show that the method has some discoverable sources of error, that the instruments likewise contain certain inherent sources of error, and finally that however care-

fully the effects of the discoverable sources are removed, some undiscovered or uncorrected sources still remain, since successive equally careful repetitions of the same measurement yield numerical results which are more or less discordant in the last one or two places of significant figures.

Example I (A), page 9.

The existence of this discordance just referred to proves that the errors from the various sources are not constant, at least that some of them are not,—a fact which we know to be true for some of the discoverable sources. And the general rule for the variation doubtless is that under given conditions the error from any given source has a certain average magnitude about which it varies more or less, being sometimes greater sometimes smaller than that amount. It is therefore reasonable, and will be found convenient, to regard the error from any source as made up of two portions, a constant part, viz. its average value, and a variable part. Of course either of these parts may be wanting in any given instance.

Errors of Single Observations.—The actual error of any given single observation is obviously the algebraic sum of all the individual errors from the several sources which affect the quantity. As these individual errors have each a constant part and a variable part, so the error of a single observation will be made up of two parts. These will be a constant portion which is the algebraic sum of the constant parts of the individual errors, and a second portion which will vary in different observations and which is, for any observation, the algebraic sum of the variable parts of the individual errors as they existed at the moment when that observation was made.

Variable Part.—Considering first this part of the error, we can at once see that, if we make a series of observations under sensibly the same conditions and take the average, the result will be partly free from the effect of the variable parts of the error. For each varying error will tend to make the result at one time more or less too large, at another too small,—a kind of fluctuation which the process of averaging tends to eliminate. That the arithmetical mean removes the variable parts

of the error better than any other function, will be shown more explicitly later. It is easy to see that averaging can never effect a complete removal; the elimination being, however, more nearly complete (in proportion to $\sqrt{n}$) as the number, n, of observations is greater. For the sum total of the positive parts of the variation will naturally be unequal to that of the negative parts, and so long as the number is small the inequality will be considerable, but they will become more and more nearly equal as the number increases. At best, however, there will always remain a residual variable error, and in discussing the correctness of the result this must be taken into account.

Example I (B), page 9.

Constant Part.—Considering next the effect of the constant parts of the errors from the various sources, we see that their resultant, viz., the algebraic sum of all these constant parts, will itself necessarily be of the same amount in each and all the single observations taken under the same conditions. The process of averaging a series will therefore do absolutely nothing toward the removal of this resultant of the constant portions. The mean will contain an error of which the constant portion will be identical with that of each single observation in the series. The name ***"constant error"*** is therefore well applied to this resultant constant portion of the errors.

Elimination of Constant Error.—Of the constant portions of the individual errors going to make up this resultant constant error, some will be positive, some negative, and the magnitudes will be various. They will therefore in part annul one another; that is, the resultant will not in general be as large as the arithmetical sum of all the component parts. If the sum of the positive parts exactly equalled that of the negative parts, the elimination would, of course, be complete, and the constant error of the single observation and mean would be zero. But this condition would naturally be highly exceptional. The constant error is in fact exceedingly difficult of removal, and often proves to be of surprisingly large amount in spite of most painstaking efforts for its elim-

ination. For any specific method or apparatus will have its own characteristic set of errors, of which some will be predominant and will determine a constant error more or less large in the results obtained. Another method will on the other hand be characterized by a different set of sources of error and will have a different constant error. Observations taken under diverse conditions with the same method will often also have differing constant errors. And finally, different observers will show different "personal equations." Thus, results obtained by changing methods, apparatus, observers, or other conditions will materially differ in the sources and amounts of their constant errors. The greater the number and the more complete the variety of the changes, the greater becomes the diversity of the sources of error, and consequently the more complete the elimination of their effects by taking a general mean (with due respect to weights) of the various results. It is only through this repetition by independent methods that we can gain confidence as to the real accuracy of a result; and the more radically distinct the nature of the methods employed, the more valuable the check. Even a single reliable check greatly enhances the value of a result. It is evident that we can obtain no numerical measure of the amount of the constant error of any result. Such a measure would imply the knowledge of the true result, which is of course always unknown.

Example I (*C*), *page* 10.

Corrections.—In as much as each method and apparatus has its own characteristic sources which determine its constant error, it is obvious that in using any method we must do what we can to reduce the effect of those sources to a minimum. For this purpose we must study the method and instruments as thoroughly as possible in advance, to discover all possible sources of error. We must then arrange the work to remove as many as possible of these sources wholly or in part, and we must evaluate the effects of those not removed and eliminate them by the application of the corresponding "corrections" thus determined.

Example I. (See pages 6, *7 et seq.*)—Take as an illustration so simple a measurement as that of the distance between two points by means of a steel tape.

(*A*) There are easily discoverable such sources of error as these:

(1) Error in numbering of tape;
(2) Irregular spacing of divisions;
(3) Incorrect unit, i.e., foot not standard in length;
(4) Bends in tape;
(5) Sag of tape;
(6) Stretch of tape·
(7) Error in setting zero of tape at starting point;
(8) Error of estimation of fraction of division at finishing point;
(9) Temperature not that for which the tape was graduated.

Besides these sources there are doubtless many others of greater or less effect, some of which might possibly be discovered by further study, but many of which are at present obscure.

Successive measurements of the same distance, especially if this be long and if the fraction of an inch to which readings are taken be small, will show discordances of greater or less magnitude.

(*B*) *Variable Part.* Errors 5, 6, 7, 8, 9 would vary in amount from time to time and between different readings, and would therefore have variable parts. Each would tend to make the single results sometimes larger, at other times smaller, and by irregular amounts. Thus in the average result of a series of observations the variable parts of the error from any single source would in part annul itself. Also in any single observation the sum of the negative variable parts of the errors from all sources would offset in part the sum of the positive variable parts more or less completely, but seldom wholly.

(*C*) *Constant Error.* The errors 1, 2, 3, and 4 would be constant for any given distance; also, 5 and 6 will clearly

be liable to have some constant portion, as also would 9 under some circumstances. These together will make up the constant error. Some will be of one sign, some of the other, so that they will in part neutralize, but cannot be expected to wholly do so. The separate constant portions are the same in all single observations. Hence the constant error will be the same in each single reading and in the mean result.

The part of the constant error due to 5 and 6 may be largely removed by stretching the tape by a spring-balance or other means so that it is always under the same tension. A correction for the amount of sag can then be made, and the stretch will be nearly the same at all times. The error from 9 may be in part removed by measuring the temperature of the tape at various points along its length and correcting for the expansion due to the difference between the observed temperature and that at which the tape is correct. There is liability in this correction to a residual constant error due to uncertainty as to the value of the coefficient of expansion for the metal employed, and due to constant errors in the thermometers. There is liability to variable error from the thermometers being too far apart or improperly located to give the true mean temperature of the tape; also from the variable errors of the thermometers themselves.

It is clear, upon reflection, that the errors in the above measurement, especially the constant errors, are largely peculiar to the method. If the distance were to be measured by a rod, or by a base-line apparatus, or by stadia wires in a telescope, each of the results would be characterized by a different set of errors, of which some might be common to all. The mean of the results of such different methods would be certainly more reliable than the poorest of them, by the natural annulling of the different classes of errors.

Determinate and Indeterminate Errors.—All sources of error which are discoverable and which may be removed or may have their effects more or less completely allowed for by corrections will here be classed as *determinate* sources. The corresponding errors will be referred to as determinate errors.

Some errors are determinate as to their nature only, others as to sign, others as to both sign and magnitude.

On the other hand, all sources which are either undiscoverable, or whose effects cannot be properly determined and allowed for, will be classed as *indeterminate* sources, and the corresponding errors, as indeterminate errors. This class will contain not only those which are undiscoverable, but also the residuals of determinate errors. Both determinate and indeterminate sources are inevitably present in every direct, and therefore also in every indirect, measurement.

Residuals.—In general the processes for the elimination of the determinate errors, whether by the removal of their sources or by corrections, accomplish this object only approximately. There are perhaps a few cases in which the source of a determinate error can be wholly removed, or at least to a far greater extent than is demanded. For instance, in the constant π we may retain so many places of figures that the error from rejecting the rest may be utterly insignificant. Ordinarily, however, the source cannot be removed, but its effect can be lessened so as to be small or negligible. For instance, the individual weights of a set can perhaps be adjusted so accurately that their errors are negligible for a purpose in hand; or the arms of a balance may be made so nearly equal that the error is negligible. But in all such cases there remains an error more or less small which enters into the result and which will be called a residual.

A residual may be insignificant, but this requires proof; and the proof can only be arrived at, in general, by a direct measurement of some kind, such as a comparison with a standard, or a measurement of some ratio. For instance, the weights of the set can be assumed to have a negligible error only after each has been weighed against a standard or tested by some equivalent process. This weighing will be made only to a certain grade of accuracy, and will therefore itself leave a residual error. Similarly the ratio of the arms of the balance must be determined by the usual process of balancing and interchanging equal masses. Therefore this also is a direct measurement,

and will of course be made only to the limit of accuracy fixed by the sensitiveness of the balance, and will leave a residual error.

The process of evaluation of a correction is in general also only an approximate one, and consists usually of a direct or indirect measurement carried out only with a certain degree of accuracy leaving a residual error. Thus in the foregoing examples the weights might be adjusted less closely than was demanded for the work in hand, and the corrections to be applied might be evaluated by weighing against standard weights and thus determining the error. But this weighing would be a direct measurement, and would be carried out to an accuracy limited by the sensitiveness of the balance or by some other conditions. A corresponding residual error would therefore be left after the correction was applied. Similarly, the ratio of the balance arms not being close enough, it might be allowed for by measuring the ratio and applying a correction. This again would leave a residual error.

Other processes of correction exist, such as the correction for the eccentricity of a circle by reading two verniers 180° apart, and averaging. This is a type of certain mathematical corrections, and these also are usually only approximate, being close enough when the errors are small, but nevertheless leaving residual errors.

In brief, then, most processes for the elimination of determinate errors leave residual errors behind. Also, most such processes involve direct measurements, and the statements already made or to be made respecting direct measurements apply to them. The numerical measure of the residuals will in general therefore be of the nature of precision measures of direct measurements which will presently be discussed.

From this it is obvious that if it were necessary in an investigation to work out from the beginning every detail of a research, establishing all standards, ascertaining all corrections, developing every process employed, the labor would be enormous—as, indeed, it often is. But fortunately the progress of experimental science has provided instruments, pro-

cesses, methods, and results of known accuracy, which may be appropriated in any desired manner in more complex investigations.

Accuracy or Error of Results.—By the accuracy of a result we mean its freedom from error. The real measure of the accuracy of a result is therefore the error of that result. Thus if we knew that a result had an error of 2 per cent we should say that it was *accurate to* 2 per cent, or we might say that its accuracy was 98 per cent. The latter phrase, although more exact, is less common and convenient than to say that the accuracy was 2 per cent. Thus if a result were 24.967 metres, and were known to have an error of 0.025 metres, we should say the result was accurate *to* 0.025 m. or *to* 0.1 per cent; or we might say that it had an accuracy *of* 0.1 per cent, although that phrase would be less precise than to say that it had an accuracy of 99.9 per cent.

But it is clear that we can have no numerical measure of the constant error of a result, whether that result be a single observation, a mean of a series by one method, or the mean of results by a large number of methods. For as the error is the amount that the measured result differs from the true value, such a measure necessarily implies that the true value of the quantity is known, which is never the case.

Yet it is of the utmost importance that we should be able to form some *estimate* of the accuracy or of the error of the result, and that this should be expressed numerically, so far as possible. How such an estimate is arrived at will be here indicated, and just what the measure is will be more explicitly stated in a later paragraph.

There are only two things upon which this estimate can be based, in the case of the result of a series of observations by a single method, viz.:

(1) The degree of care exercised in the study and removal of the determinate errors.

(2) The concordance, or rather the discordance, between the single observations of the series.

Of the first, we have a partial numerical measure in the

measure of the residuals of the determinate errors, but this is only partial. A judgment as to the sources of error which have been overlooked or neglected is essential, but this cannot be given a numerical expression.

Of the second, the numerical measure is the "deviation measure" to be presently described.

The deviation measure and residuals can be combined to give the "precision measure" which is the final numerical measure to which we are brought in forming our estimate, and of which the significance will be stated later.

Thus the most that can be done in forming an estimate of the error of the result of the mean of a series of observations by a single method is this: the precision measure of the result is calculated, giving a partial numerical measure; and a judgment is formed from an inspection of the method as to whether any constant error comparable with the precision measure probably exists in the result.

If results by several different methods, etc., are available, the best representative value (weighted mean) can be obtained from them, and their concordance will give us a further partial indication of the correctness of that value.

It becomes necessary, therefore, to discuss the meaning of the terms, and to fix upon certain points respecting deviations and their measure, and the precision measure.

Deviations.—Suppose any number, n, of direct measurements or observations of a quantity to have been made with equal care, and under apparently identical conditions. Let $a_1, a_2, \ldots, a_n$ represent the separate results. Let A represent their arithmetical mean or average. Then the differences of these from the mean will be given by

$$d_1 = a_1 - A, \quad d_2 = a_2 - A, \quad \ldots, \quad d_n = a_n - A.$$

These differences will be called the *deviations* of the single observations from the mean. They are the effects of the *variable parts* of the errors affecting the measurements. They are not the *errors* of a_1, a_2, etc.; for errors are the discrepancies

between observed and *true* values. But in this as in all cases the true value is unknown, and the deviations are merely the differences from the mean value A, which is selected as being the best representative value, but may differ much from the true value. Thus the deviations measure only the variable part of the errors and give no clue whatever to the constant parts.

General Law of Deviations.—If the number, n, of observations in the series be very great (to eliminate exceptional irregularities), it is found as the result of the study of actual series of observations that the deviations follow a definite law, both as to sign and magnitude. This law is apparently the same for all kinds of measurements which are affected by a large number of sources of error, and may be called the general laws of deviations. Special laws arise in certain cases, as will be further indicated. The general law may be approximately stated in words thus: Positive and negative deviations of any given magnitude occur with equal frequency; small deviations are more frequent than large ones; very large deviations occur very seldom. The law is more exactly expressed by the equation

$$y = ke^{-h^2x^2},$$

where y = frequency of occurrence of deviation whose magnitude has any assigned value x, and where k and h are constants, and ϵ is the base of the Naperian system of logarithms.

This expression was deduced by Laplace by an *à priori* mathematical process as showing the probability of occurrence of an error of any given magnitude when the error was not of simple origin, but was produced by the algebraic combination of a great many independent causes of error, each of which, according to the chance which affects it independently, might produce an error of either sign and of different magnitude. Applied to actual series of observations it is found to sensibly coincide with the distribution of their deviations. This exponential equation may then be held as representing the general

law of distribution of deviations, being in accord both with the theory of probabilities and the results of experience. It is sensibly exact when the number of observations is large. When the number is small, the distribution can follow this law only roughly, but no other law would be more closely followed. The approximation with which the series of observations is represented by the law is then greater the larger the number of observations in the series.

Mean: Best Representative Value.—In a large series of equally careful observations of the same quantity, under the same conditions, the variable parts of the errors will be sensibly eliminated by averaging the results, that is, by the employment of the mean as a representative value. The law of deviations already stated shows that to be true, and as this law has been arrived at by an application of the theory of probabilities and confirmed by the results of specific as well as of general experience, the use of the arithmetical mean as the best representative value in such a large series can be considered as in accord both with the theory of probabilities and with practical experience. But its employment is, however, justifiable not in in large series only, but in small ones as well. For although the reliability or degree of probability of the mean in a small series will be less than in a larger one, yet the mean has a greater probability even in a very small series than any other representative value which can be indicated.

We are accustomed to think of the mean as being more reliable in proportion to the square root of the number of observations in the series, but we must avoid attaching undue weight to this numerical relation when the number of observations is very small, as for instance when not exceeding five or ten. A similar caution should be urged respecting all applications of the methods and rules of least squares when n is small, although the use of the methods in such cases is fully justified by the fact that they give the best results obtainable.

Deviation Measure, Average Deviation.—The magnitudes of the deviations in a given series, although giving no indication as to constant errors, do furnish a measure of the variable

parts of the errors, since it is to these that they are due. But where the number, n, of observations is not very small, mere inspection does not readily give a definite idea of the magnitude of the deviations; moreover for many purposes of calculation it is necessary to have a single number to represent them. The simplest method of obtaining such a number is to take the arithmetical mean of the deviations without respect to sign, that is, with regard to magnitude only. This quantity will be called the *average deviation of the single observation*, and will be denoted by *a.d.* Thus

$$a.d. = \frac{d_1 + d_2 + \ldots + d_n}{n} = \frac{\Sigma d}{n}. \quad . \quad . \quad . \quad [1]$$

This, being obviously a measure of the deviations, will be called the *deviation measure of the single observation.* It gives, at least approximately, the measure of the *variations* of the resultant indeterminate errors of the *individual* observations. It shows also that in the given series the observations differ on the average from the mean by this amount; and we may infer or predict that more observations taken under the same conditions will on the average differ from this mean by about this amount.

If we have two series of observations consisting of a different number of observations n_1 and n_2, respectively, all taken under the same conditions and with equal care, then the mean result of the series for which n is greater will be more free from the effects of the variable parts of the errors. The principle of least squares, based upon the general law of deviations, shows that the reliability in this respect will be in proportion to the square roots of the number of observations respectively, that is, as $\sqrt{n_1} : \sqrt{n_2}$. Hence we may say that the mean result of a series of observations all made under the same conditions and with equal care is more free from the effect of the variable parts of its errors in proportion to $\sqrt{n}$, that is, to the square root of the number of single observations

from which it is computed. Hence the deviation measure of a mean result would be that of the single observation divided by $\sqrt{n}$. Thus, using the average deviation, the deviation measure of the mean result would be

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{\Sigma d}{n\sqrt{n}}. \quad \ldots\ldots\ldots \quad [2]$$

This will be called the *Average Deviation of the mean.* It measures the effect upon the *mean result* of the average of the variable parts of the errors entering into the single observations, and obviously bears the same relation to a mean result that *a.d.* does to a single observation.

Example II.—Suppose 9 separate observations were taken of the distance between two points with the results headed *a* in the table. The mean result to be used would then be $A = 16.2799$. The deviations would be found by subtracting A from the values in column *a*, and are given in column *d*. The deviation

a	*d*
cm.	cm.
16.224	− 0.006
.233	+ 3
.221	− 9
.230	± 0
.234	+ 4
.243	+ 13
.225	− 5
.231	+ 1
.228	− 2
9) .269	9) .043
$A = 16.2299$	$0.0048 = a.d.$

$$\therefore A.D. = \frac{0.0048}{\sqrt{9}} = 0.0016 \text{ cm.}$$

measure of the single observation would be the *a.d.* = 0.0048. The deviation measure of the mean would be the *A.D.* = 0.0016. This would show us that if we made use of the mean result 16.2299 in any work, the deviation measure to be used would be 0.0016. But if at any other time a single observation only were made of the same distance under apparently identical conditions, and that single result were to be used, the deviation measure which must be used in connection with it would be the *a.d.*, viz. 0.0048.

The relative significance of the *a.d.* and *A.D.* may be put in another way also. If we wished to compare, as to concordance, a number of mean results taken at different times but under similar conditions except as to number of observations, we should use the *A.D.* of each mean. If we were comparing the relative precision of the single observation in one of these series with that in any other one we should make use of the *a.d.*

The abbreviation *d.m.* will be occasionally written instead of the full term "deviation measure." It will be understood to denote any deviation measure, viz. the *a.d.*, *A.D.*, or any of those described below, according to the context.

The deviation measure is often called the "precision measure," * but the latter term is reserved for another use in these pages.

It is essential to note exactly the significance and limitations of the deviation measure. It does not tell us that the result, whether a single observation or a mean, is in error by this stated amount (e.g. the *a.d.* or *A.D.*), but merely that *the variable parts of the errors produce a variation of that average amount* in the results. By the law of distribution of these deviations we know that the deviation of any individual observation may be many times the *a.d.*; or of a mean result, many times its *A.D.* In fact that law shows that the chances

* This usage was adhered to in the printed Lecture Notes prepared upon this subject, but experience has shown that the change to deviation measure is desirable.

that the deviation will assume certain specified magnitudes are those given in this table.

0	100.
½ *a.d.*	69.
a.d.	43.
2 *a.d.*	11.
3 *a.d.*	2.
4 *a.d.*	0.1

Column second gives the percentage of the whole number of observations which would have a deviation greater than ½ *a.d.*, *a.d*, 2 *a.d.*, etc. Thus in any series sufficiently large to fulfil the conditions under which the general law of deviations holds, 43 per cent of the single observations would have a deviation greater than the average, 11 per cent only (i.e. about one in ten) greater than twice the average, and only one in one thousand greater than four times the average. Thus in the foregoing example, where the *a.d.* was 0.0048, we may say that the chances are 43 to 57, or roughly about even, that any single observation is affected by the variable parts of the errors to an extent of ± 0.0048 units. The *A.D.* of the mean of that series is 0.0016, so that we may say of the mean that the chances are nearly even that it is thus affected to the extent of about ± 0.0016 units.

Places of Figures in d.m.; and Negligible Amounts.—In the numerical value of any deviation measure, two and only two significant figures should be retained; as was done in the above example. Any single change in the measured quantity, *a*, due to whatever cause may be regarded as negligible when not exceeding $\frac{1}{10}$th of the deviation measure of the quantity. Therefore *a* should be carried out to the place corresponding to the last significant figure of the *d.m.* Similarly any change in the *d.m.* is negligible when not exceeding $\frac{1}{10}$*d.m.*

The fractional and percentage deviations, *d.m./a* and 100 *d.m./a* (see page 29), should also contain two and only two significant figures; and any fractional change is negligible in

them when not exceeding $\frac{1}{10}$th of their values. Similarly any fractional change in the measured quantity a is negligible when not exceeding $\frac{1}{10}$ *d.m./a.*

These statements may be justified as follows: Taking the numbers used in the above example, let 16.2299 denote a mean result of a direct measurement, and 0.0016 its deviation measure. The latter shows that the number 16.2299 is uncertain by 16 units in the sixth place of significant figures. A change corresponding to $\frac{1}{10}$*d.m.* would be 2 in this sixth place, already uncertain by 16. It is therefore clear that such a change is immaterial and may be regarded as negligible. This change of 2 being negligible in the number an equal change would be negligible in the *d.m.*, and, as this is 10 per cent. of that number, a change in *d.m.* of $\frac{1}{10}$*d.m.* is negligible. Obviously also the figure corresponding to $\frac{1}{10}$*d.m.* will always be in the second place of significant figures, so that if we always retain that place and always reject all figures beyond that place in *d.m.*, we shall never introduce by that process an error exceeding this limit into the *d.m.* Hence two places of significant figures in *d.m.* are enough.

This limit of $\frac{1}{10}$*d.m.* as the negligible amount is an arbitrary selection. A larger or a smaller amount might have been chosen as the limit, but experience shows this to be both convenient and suitable in practice. Yet in rather rough work a larger limit may be used, and for such work the *d.m.* need be retained only to one place when not less than 5 in that place. For instance, if the above example represented rather rough work, the *a.d.* might be written 0.005 instead of 0.0048, but the *A.D.* would rarely be written 0.002 instead of 0.0016.

By inspection it is easy to see from these statements that the numerical result should in general be carried out to the place corresponding to that of the second significant figure of the *d.m.* of the quantity. Thus the number 16.2299 should be carried out to the sixth place of figures. This statement is true whether the result is a mean or a single observation, the *d.m.* being in the first case the *A.D.*, in the second the *a.d.* Inspection of the data in any case will usually show us what place

will correspond to the second of the *d.m.* even in advance of the exact computation of that quantity.

It is obvious that if $\frac{1}{10}d.m.$ is negligible, $\frac{1}{10}d.m./a$ will be also, for both are the same part of a. Similarly if $d.m.$ must be carried to two places to correspond to this limit, $\frac{1}{10}d.m./a$ must also be carried to two places. The same is, of course, true for the percentage precision.

In computing the deviation d by subtracting A from a_1, etc., it is usually unnecessary to retain for this part of the work more places in A than are given in a_1, a_2, etc., in the observations. Thus in the foregoing example, to find d_1, etc., we use 16.230 instead of 16.2299. If, however, the values of d are very small, the largest value not exceeding perhaps 2 or 3 units in the last place, then it is better to retain the full number of places in A or at least one more than in the values of a. For instance in the example if 16.233 had been the largest and 16.228 the smallest value of a and the mean had been 16.22 993 the deviations would have been formed using 16.2299. It would be useless, however, to retain 16.22 993 for this purpose, although it might be proper to retain it for other uses. It is, however, to be noted that when the apparatus gives indications which continually agree within one or two units in the last place of figures obtained by the single observation, it is delusive to hope for much gain in precision by many repetitions. Such cases often occur in practice. They usually show that the indicating part of the apparatus, whatever it may be, is not as sensitive as it might advantageously be made. Thus if in making a weighing of the same object repeatedly by the ordinary method, we find that the results agree to one or two units in the last place, e.g. to 0.1 or 0.2 mgr., this indicates that the balance is delicate enough to have a finer index or to be used by the method of swings.

Best Value of n.—The question continually arises, how many observations is it worth while to take in order to reduce the $A.D.$ of the mean? Since $A.D. = a.d./\sqrt{n}$, the gain is only in proportion to $\sqrt{n}$. But the labor of observing is in direct proportion to n. Thus to double the gain the labor must be

fourfold, to treble it ninefold, i.e. the labor is as the square of the gain. Obviously then a point would soon be reached where the labor would become excessive in comparison with the gain or with the labor involved in other parts of the work. The limit to the number n of observations to be taken must then be determined by the judgment of the observer as to when the labor becomes excessive in proportion to the gain. In ordinary work $n = 9$ is often a convenient and sufficient number, though a smaller number will frequently suffice. It is rare except in the most careful work, or in work of some special character, that n is made to exceed 25.

Other Deviation Measures.—Other quantities than the average deviation are also employed as deviation measures. In fact the most common measure is not the average deviation but the so-called "probable error." The relation between the probable error and average deviation is given by the expressions

$$p.e. = 0.84\ a.d.; \quad P.E. = 0.84 A.D., \ldots [3]$$

where $p.e.$ is the probable error of the single observation, and $P.E.$ that of the mean result. The ordinary formulæ for computing the probable errors from the square root of the sum of the squares of the deviations possess no real advantages over the above, while far more laborious. The probable error, $p.e.$, is merely a deviation of such magnitude that there are, in a large series, just as many deviations greater as less than it; or in other words, such that in the series there are just as many observations having values lying between $A + p.e.$ and $A - p.e.$ as outside those limits; so that it is an even chance whether any observation taken at random will have a deviation greater or less than $p.e.$

The use of the "probable error" is objectionable partly because of its more artificial character, partly because of the greater labor of computation, but chiefly because the term is seriously misleading. It is, in the first place, not an *error* at all, but merely a *deviation.* Neither is it a "probable" value in the ordinary sense, as it is more probable, that is of greater

frequency of occurrence, than any given larger deviation, and less probable than any smaller one. Its use leads almost inevitably to a fallacious impression as to the real accuracy of results, and tends to promote negligence as to the constant errors which are of far more serious importance. For a reader meeting a result stated to have a small probable *error* is liable, unless unusually upon his guard, to receive at once an impression of accurate work and to have his attention diverted from other points upon which the reliability of the work depends to a greater extent. And an observer, with the natural tendency to confidence in his own work, is even more easily misled by the term "probable error." The term "average deviation" tends, on the contrary, rather to call attention to the true character of the quantity; and by the use of "error" solely in connection with constant errors, attention is the more strongly directed upon these. To a competent and experienced observer this discrimination in terms is unimportant, but it is by no means so to the beginner.

It may be remarked that the numerical difference between the average deviation and the probable error is negligible in almost all work if we follow the limit already set (viz., $\frac{1}{10}$ *a.d.* or $\frac{1}{10}$ *A.D.*). It is also of course true that in all the formulæ developed, the probable error may be inserted to replace the average deviation, if desired, a little attention being given to insure consistency.

Special Laws of Deviations.—Besides the foregoing general law there are other laws which the deviations follow in certain cases. Of these special laws the only one with which we are concerned is that occurring when any deviation between the limits $+a$ and $-a$ is equally likely to be obtained, i.e., where all deviations between these limits have an equal frequency. It is easy to see by inspection that the average deviation under this law must be $\frac{1}{2}a$.

This is the law according to which the deviations occur when tenths of a division are estimated by the eye. With moderate practice, divisions of not less than half a millimeter can be read to tenths with the unaided eye so that the estima-

tion shall always give the nearest tenth, that is, so that the error or deviation shall not exceed + 0.05 or − 0.05 mm. Now as the point to be read is equally likely to lie anywhere along the scale, its actual distance from the estimated tenth is equally likely to be anything within these limits. Thus the *a.d.* of a single estimation will be 0.025 or $\frac{1}{40}$th of a division. Experience demonstrates that this limit is reached without difficulty, and often exceeded where an attempt is made under good conditions to estimate twentieths instead of tenths.

Precision Measure of Result.—Let *d.m.* denote the deviation measure of the result, viz. the *a.d.* if the result be a single observation, and *A.D.* if it be a mean. Let r denote a residual (page 11) left by the elimination of a determinate error, $r_1, r_2, \ldots, r_p$ being the respective residuals from p determinate errors.

Then the term *precision measure*, *p.m.*, will be hereafter used to denote the quantity δ given by the expression

$$\delta^2 = d.m.^2 + r_1^2 + r_2^2 + \ldots + r_p^2 \quad . \quad . \quad . \quad . \quad [4]$$

The precision measure of a direct result includes therefore both the deviation measure and the residual effects of all determinate errors, so far as they can be numerically expressed. It is thus the best and only *numerical* measure obtainable of the accuracy of that result taken by itself, but it fails, of course, to indicate anything more respecting the constant errors than to imply that so far as determinate these have been removed.

If the residuals r are all negligible as compared with the *d.m.*, then the precision and deviation measures coincide. The law of accumulation by squares from which the above expression for δ is deduced is based on the principle of least squares, and will be further discussed in late sections.

The term *precision* is used intentionally rather than *accuracy* in the foregoing paragraphs. A distinction between the denotation of these terms will be maintained. Accuracy will be used only when attention is distinctly directed toward the constant as well as the variable errors. An estimate of the

accuracy of a result thus involves a discussion of possible constant errors. The precision measure although implying when properly used that no determinate constant errors remain, does not call for a discussion of the constant errors. A result might be precise, and yet contain a large *unknown* constant error, but it would not then be accurate, in the sense in which these terms are here employed.

To Make Residuals Negligible in P.M.—One or more of the residuals, r, in the expression for *p.m.* may become negligible.

Criterion.—The criterion is as follows: Any single residual may be regarded as negligible when

$$r \leqq \tfrac{1}{3} d.m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [5]$$

Any number, q, of the residuals are simultaneously negligible when the square root of the sum of their squares is $\leqq \frac{1}{3} d.m.$ For instance, r_2, r_3, r_4 are simultaneously negligible when

$$\sqrt{r_2^2 + r_3^2 + r_4^2} \leqq \tfrac{1}{3} d.m. \quad . \quad . \quad . \quad . \quad . \quad [6]$$

A simple though less general criterion for this case is that each neglected residual must not exceed

$$\tfrac{1}{3} d.m. / \sqrt{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [7]$$

This is based on the assignment of equal effects discussed in the next section.

Demonstration.—It has been shown that any change which affects the deviation measure by 0.1 of its amount or less is negligible, and for similar reasons the same is obviously true for the *p.m.*

Suppose first that there is but one residual, r_1, what value may r_1 have consistently with the above limit? For this case we shall have, respectively,

$$\delta^2 = d.m.^2 + r_1^2$$

and

$$\delta_1^2 = d.m.^2,$$

for the true value of δ, and for the value when r_1 is omitted. Then, in order that r_1 may be negligible, $\delta - \delta_1$ must be $\leqq \frac{1}{10}\delta$.

We have then

$$\delta = \sqrt{d.m._1^2 + r_1^2}, \qquad \delta_1 = d.m.;$$

$$\therefore \delta - \delta_1 = \sqrt{d.m._1^2 + r_1^2} - d.m.$$

But $$\delta - \delta_1 = \tfrac{1}{10}\delta = \tfrac{1}{10}\sqrt{d.m.^2 + r_1^2};$$

$$\therefore \sqrt{d.m.^2 + r_1^2} - d.m. = \tfrac{1}{10}\sqrt{d.m.^2 + r_1^2}.$$

$$\therefore \tfrac{9}{10}\sqrt{d.m._1^2 + r_1^2} = d.m., \text{ and}$$

$$r_1^2 = 0.23 d.m.^2; \qquad \therefore r_1 = 0.48 d.m. \quad . \quad . \quad . \quad [8]$$

Hence r_1 will be negligible when less than $0.48 d.m.$ The limit which will be here employed, however, will be $\frac{1}{2} d.m.$ This is adopted as being a convenient number and as making a safer allowance when the number of residuals is small, and the approximate formula of squares consequently less reliable.

Next, if there are p residuals, we may easily show by the same process that the actual limit for r_1 would be

$$r_1 \leqq 0.48 \sqrt{d.m.^2 + r_2^2 + \ldots + r_p^2}. \quad . \quad . \quad . \quad [9]$$

But instead of using this exact expression we may employ the same limit, viz., (5), in this case as when there is only one residual. For the latter is evidently a smaller limit, and therefore safe, and is more convenient.

The criterion (6) stated above for the simultaneous negligibility of several residuals is easily deduced by the same process.

This limit at which the effect of the residuals becomes negligible is unfortunately beyond attainment in many cases in practice. With a considerable proportion of all direct reading instruments the corrections cannot be determined much more closely than the *a.d.* of the ordinary readings.

Best Value of the Residuals: Equal Effects.—The following case is of frequent occurrence. Given a direct measurement in which there are p residuals not negligible but whose

joint effect must not exceed a stated limit l; to what value or limit is it most advantageous to determine each residual. This is essentially the same problem that is discussed fully later in a section headed "Equal Effects" for the components of an indirect measurement. Only the result therefore will be here stated. The relation of the residuals to l is

$$l^2 = r_1^2 + r_2^2 + \ldots + r_p^2. \quad . \quad . \quad . \quad . \quad [10]$$

The best value for the residuals is given approximately by

$$r_1 = r_2 = \ldots = r_p = \frac{l}{\sqrt{p}}, \quad . \quad . \quad . \quad . \quad [11]$$

that is, they must all be equal and therefore of "equal effect" on l and thus on *p.m.*

Although this rule affords the best solution to use as a starting point, it is only approximate and not necessarily final. The exact values of r which would be best in every case would, of course, be those which would give the stated value of l with the least labor. We cannot, however, readily obtain a solution on this basis. The values given by (*11*) will comply with this requirement only when there is equal difficulty in obtaining each of them. If some of the eliminations are more difficult than others, then the best values for the residuals of the more difficult would be larger, those for the less difficult being therefore correspondingly smaller; but the departure from equality must always be small. The residuals of the more difficult eliminations should rarely be allowed to increase to twice the value for equal effects, and if one or more of the residuals is increased the others must be correspondingly diminished in order that the limiting value of l shall not be exceeded. It must be left largely to the judgment of the observer to determine what departure shall be made from the condition of equal effects. The further statement made in the paragraph referred to should be consulted.

The usual case is where the result is desired with a stated *p.m.*, and the *d.m.* of the apparatus is fixed or is determinable

by a preliminary trial. The numerical value of the limit l of the combined effect of the residuals will then usually be determined by the expression

$$l^2 = p.m.^2 - d.m.^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad [12]$$

In case the *d.m.* is not known in advance it must be found at the outset by making a preliminary series of observations. Now the final result will generally be a mean, so that its *d.m.* will be the *A.D.* To find this from the preliminary observations, we must know the number n of observations which are to enter into the final mean. But this cannot be fixed upon at this stage, so that it is necessary to assume a number. Ordinarily it will be on the safe side to assume $n = 25$, and thus find $d.m. = A.D. = a.d./\sqrt{25}$ from the preliminary value of *a.d.* If this cannot be done it is sometimes sufficient to estimate a value of *d.m.* for a rough preliminary calculation of the limits of r.

Fractional Deviation. Fractional Precision.—Let a denote a single observation and *a.d.* its deviation measure; then $a.d./a$ is the fractional deviation—more properly fractional deviation measure—of the single observation. Similarly $100\ a.d./a$ is the percentage deviation. If A be the mean of a series of values of a, and *A.D.* its deviation measure; then $A.D./A$ will be the fractional deviation of the mean, and $100\ A.D./A$, the percentage deviation. These quantities are to be carried to two places of significant figures only since *a.d.* and *A.D.* are so, for reasons already stated. Therefore a very rough value (two places of significant figures) is all that is needed for a or A,—an important point in some applications of the methods of this subject. For this reason we obviously may substitute A for a in the above formula.

Similarly also if *p.m.* is the precision measure of any quantity a whether a mean or a single observation, $p.m./a$ is its fractional precision and $100\ p.m./a$ is its percentage precision. The above remarks as to significant figures also apply here.

Mistakes.—Errors due to such causes as recording an observed number incorrectly, counting up weights wrongly, incorrect numbering of a scale, faulty arithmetical work, are classed as mistakes. In any work under ordinary conditions where the number of observations is not great, these mistakes do not fall in with the deviations, and are not eliminated by averaging. They can only be detected by careful inspection and by check observations or computations.

Criterion for Rejection of Doubtful Observations.—It often happens that among several single observations taken under the same conditions and with equal care one deviates quite widely from the rest, so widely that the possibility of its containing some mistake suggests itself. In such a case, inspection of the records or of the work may show conclusively that a mistake did occur, and may possibly point out its exact amount. If the existence of the mistake is thus established, the faulty observation should in general be cancelled and wholly rejected; but if inspection shows positively its exact amount, the mistake may be rectified. It is always better to reject an observation than to correct it unless the mistake is perfectly obvious and its amount certain beyond question. Usually, however, the cause of the wide deviation is not apparent, no obvious mistake being discoverable, and the question arises as to whether the observation should then be rejected or retained. Mathematical criteria based on the theory of probabilities have been given by Peirce, Chauvenet, and others to decide this question, in any given case, but they are somewhat complicated in application, and a much simpler one is sufficient for ordinary work.

Criterion.—Take the mean and the *a.d.* of the observations, omitting the doubtful one. Find the deviation, *d*, of that one from the mean. Then reject the observation if $d > 4a.d.$

This limit is arbitrary and might perhaps be made narrower to advantage. It is not based upon any supposition that an observation with a greater deviation than $4a.d.$ necessarily or even presumably contains a mistake. On the contrary, if the law of deviations is followed, observations with a greater de-

viation than this will sometimes although infrequently occur, the frequency of the deviation $4a.d.$ being only 1 in 1000. The basis of any such criterion is rather this: That inasmuch as the number of observations in the series is always comparatively small, the large infrequent deviation would have undue influence if allowed to remain; so that the mean taken after rejecting it is likely to be more reliable than that which would result if it were retained.

Something must also be left to the judgment of the observer as to the propriety of making a rejection; and he is especially entitled to exercise an autocratic power in this regard if he has good reason for even suspecting that some excessive or extraordinary cause of error has influenced any given observation. If this is the case the observation ought invariably to be rejected, for one doubtful observation may vitiate a mean by a greater amount than can be compensated by many good ones.

There is a tendency, especially among inexperienced observers, to become biassed by the first one or two readings of a series, and to reject, without recording it, any later one which does not closely accord with these, tacitly assuming it to be faulty. This is an essentially vicious practice which cannot be too carefully avoided. Other things being equal the later observations are entitled to greater rather than less weight than the earlier ones, and no result should be rejected without sufficient warrant. Above all things, the integrity of the observer must be beyond question if he would have his results carry any weight, and it is in the matter of the rejection of doubtful or discordant observations that his integrity in scientific or technical work meets its first test. It is of hardly less importance that he should be as far as possible free from bias due either to preconceived opinions or to unconscious efforts to obtain concordant results.

Weights.—Suppose several different independent measurements (e.g. by different methods, observers, etc.), to have been made of the same quantity. Let $a_1, a_2, \ldots a_n$ denote the results, and $p.m._1, p.m._2, \ldots p.m._n$ their respective pre-

cision measures. And suppose further that it is desired to find from these results the best representative value. Then if these precision measures give us proper indications of the reliability of the results, that is if in each case the constant error, so far as discoverable, is negligible compared with the *p.m.*, the weight p to be assigned to each result is inversely as the square of its *p.m.* Thus

$$p_1 : p_2 : \ldots : p_n = \frac{1}{p.m._1^2} : \frac{1}{p.m._2^2} : \ldots \frac{1}{p.m._n^2}. \quad . \quad [13]$$

The best representative value will then be the weighted mean, viz.,

$$\frac{p_1a_1 + p_2a_2 + \ldots + p_na_n}{p_1 + p_2 + \ldots + p_n}. \quad . \quad . \quad . \quad . \quad [14]$$

The demonstration of this proposition is given in treatises on Least Squares.

Meaning of Estimated Accuracy of Direct Result.—This can now be readily defined. When we *estimate the accuracy* of a result of a direct measurement at a given amount (e.g., if we say that it appears correct to 2 per cent.), we mean that the precision measure of the result does not exceed that amount, and that *so far as we can discover* there is no constant error which is sensible (i.e. not negligible) compared with this *p.m.*

We do not mean that the actual error of the result is of just this amount, for if we did we should correct accordingly. Neither do we mean that this is a more probable value of the error than any other. But using the average deviation as the *d.m.*, we mean that *the average effect of all the errors remaining, so far as we can discover, is of this amount and may be either + or — in sign.* This implies that if several results of this kind were to be obtained under the same conditions, the average discrepancy among them would be approximately of this amount.

Similarly when we say that a result is desired with an accu-

racy of a stated amount, we mean that the measurement is to be so made that the precision measure of the result shall not exceed the corresponding amount; and that so far as is discoverable the constant error shall not be sensible compared with this.

Forms of Problems on Accuracy of Result.—Concerning the accuracy of the result of a direct measurement by any single method, problems arise in three different forms.

First. To obtain by a proposed method the most accurate result practicable.

Second. To obtain a direct measurement of a desired quantity and have the result accurate within a specified limit.

Third. Given a completed result obtained by a stated method to estimate its accuracy.

First. *To obtain by a proposed method the most accurate result practicable.* To accomplish this the elimination of errors must be carried as far as practicable, i.e. as far as the conditions and the amount of labor which can be devoted to the work will permit. Thus all constant errors as well as the deviation measure must be reduced to the smallest practicable limit.

For this purpose, the method, apparatus, and conditions of work must be thoroughly studied to discover, as far as possible, all sources of error, with a view to their removal or to the elimination of their effects. As many as possible of these sources must then be removed by modifying the method, apparatus, or conditions of working. The magnitude of the effects of the remaining determinate sources of error must then be evaluated, i.e. corrections determined for them. Finally, a series of observations must be taken so that their average may reduce the effect of the variable indeterminate errors.

To make the result the most accurate practicable with the method, the removals and corrections must be made with sufficient exactness to reduce their residuals to negligible amounts, or rather this limit must be approached as closely as can be done without excessive labor. Also the observations must be

as numerous as may be without undue labor. The resulting precision measure and also the constant error of the result will then be as small as it is practicable to make them, and the result will therefore be as accurate as it is practicable to obtain by the method.

The limit as to what is "practicable" is determined by the labor involved in various parts of the work. What the limit shall be which may be regarded as excessive in the reduction of the residuals to a negligible amount, must be largely left to the judgment of the observer. Obviously, however, it is not to be determined by the limit of equal labor on each removal. This amount on the different removals will differ widely. It is also evident that where the removal is easy, it may well be made with a little greater accuracy, when difficult with a little less, than the exact amount corresponding exactly to the negligible limit (page 26). But to pass far beyond the limit on either side involves a *poor distribution* of the labor, and thus a less accurate result than might be reached with the same amount of labor. As to the labor to be expended in repetition of observation, this repetition is often so simple a matter that it would be foolish to neglect to take a considerable number of observations to materially increase the precision of the result; but on the other hand it is equally unwise to continue the observations beyond the point at which the labor involved becomes large compared with that in the rest of the work. After the first few observations the gain is very slow in proportion to the total labor, as already shown. It is seldom worth while to reduce the *A.D.* below the residuals, where these exist, unless it can be very easily done.

Second. *To obtain a direct measurement of a desired quantity, and have the result accurate within a specified limit.*

We must for this end fix upon a method, apparatus, etc., which will give the result with a precision measure not exceeding the specified limit. To establish the accuracy of the result it is further necessary to show that the sources of error have been so well studied that there is no indication that the con-

stant error of the result is comparable with the precision measure finally attained.

The usual order of procedure is to find the *d.m.* by preliminary trial. This with the prescribed precision measure fixes the normal values of the residuals. In general, as stated at page 28, the labor will be distributed nearly to the best advantage when $r_1 = r_2 = r_3 = \ldots = r_p = l/\sqrt{n}$.

The method, etc., would then be studied to determine whether this limit could be reached, or what limit would be practicable. Finally, the precision measure would be recomputed from the practicable values of the residuals, to see whether it exceeded the prescribed limit, in which case the method must be modified or a less accurate result accepted.

After the completion of the actual observations it is of course necessary to make a final estimate of the accuracy of the result.

Third. *Given a completed result obtained by a stated method, to estimate its accuracy.*

This question arises in reviewing results of any work after its completion. A statement will first be made which will apply to any piece of work done by another person and coming before us for inspection. It thus will apply to any published work of quantitative character. From what has been already said under the two preceding cases we can at once see that the procedure is as follows.

Study the method, apparatus, and conditions to discover as far as possible all sources of error. Ascertain whether these have been removed or corrected for. If so ascertain the measure of the residual from each. If not, all the unremoved sources constitute causes of constant error whose magnitude must be determined or estimated and considered in the final summing up. Lastly the deviation measure must be ascertained, and from this and the residuals the precision measure must be calculated. Any unremoved constant error discovered may be treated as a residual in the calculation of the corresponding precision measure if its amount or average value can be estimated.

Finally, give the resulting precision measure, and state whether, so far as discoverable, all determinate sources of error have been removed or their effects corrected so that the constant error of the result is negligible compared with the *p.m.* This forms the whole "estimate of the accuracy" of the result. It is usually well to express the *p.m.* both in units and in percentage.

Data Required to Substantiate Result.—In order to determine the reliability and value of a result it is necessary to form such an estimate of its accuracy. To do this it is essential, of course, that all the necessary data should be in hand. Any description of a method and result, as in a published paper, can then be justly criticised as materially incomplete if it does not give all the data needed for such a discussion. The importance of mentioning every source of error which has been overcome is obvious, for the presumption in case of doubt naturally is that any not mentioned have been overlooked.

The restrictions of space or time often compel the omission of such data from printed articles or from papers to be read. In such case, however, nothing can excuse the omission of a definite statement of the estimated accuracy. Failure to give the complete data can be ascribed only to urgent necessity for condensation, or to ignorance or neglect on the part of the observer, and either of the latter two cast grave doubt on the quality of the work.

That an estimate of the accuracy of his work should be carefully made by the observer and presented along with the result is but little less in importance than that the measurement should be made. The small proportionate amount of labor thus bestowed is far more effectively expended than an equal amount upon the work of observing.

Planning of Direct Measurement.—In laying out in advance any direct measurement this will have to be done under one or the other of the first two of the foregoing specifications, viz. to obtain the result by a specified method as accurately as is practicable, or to select a method and obtain a

result with a specified accuracy (which may again be of course merely the best practicable).

The method of procedure is clearly suggested by the preceding discussions, but may be summarized as follows:

(*a*) Obtain a general idea of the proposed method, apparatus, and conditions of work, or of several methods, etc., from which selection may be made.

(*b*) Make a thorough study of all discoverable sources of error, taking preliminary observations if necessary.

(*c*) Plan the sufficient removal of all determinate sources of error or determine corrections for them.

(*d*) Take the final series of observations.

Proper attention to these preliminaries is the *sine quâ non* of good results. The work involved in them often far exceeds that of taking the final observations, and is sometimes discouraging to the experienced observer as well as to the beginner. Its essential character must however not be overlooked. Without it, much time and labor is inevitably wasted and a result of inferior accuracy—often of no value whatever—is the outcome.

SOLUTION OF ILLUSTRATIVE PROBLEMS.

DIRECT MEASUREMENTS.

Example III.—*Problem.* The weight (or mass) of an object is to be measured by an equal-arm balance which reads to 0.1 mgr. The result is desired with the greatest accuracy practicable (see *Forms of Problems*, p. 33) with the given balance.

Solution.—First (see page 29) unless sufficient data are in hand, a preliminary series of weighings is made to find the approximate weight W, and the *a.d.* of the single weighings. Suppose these are found to be $W = 34$ grms. and *a.d.* $= 0.00021$ grms.

Next a study of the method and apparatus would be

made. Suppose that this shows that the work is subject to the following determinate sources of error:

(1) The balance arms may be unequal.

(2) The weights may not be standard from being irregular among themselves and by the unit not being correct.

(3) The temperature may be unequal throughout the balance case, causing inequality of balance arms, and producing air currents. This trouble may be due to the presence of the observer, to the proximity of other hot or cold objects, to the attempt to weigh a body warmer or cooler than the air in the case, or to other causes.

(4) The buoyancy of the air may be unequal on the object and weights, owing to the two having unequal volumes.

Of these (1) may be removed by readjusting the distance apart of the knife-edges, or eliminated by measuring the ratio of the arms and making a correction for it. (2) May be removed by readjustment of the weights, or may be eliminated by comparing with standard weights and applying the corrections thus determined. In both (1) and (2) if the method of removal by adjustment is adopted, we must evidently determine whether the readjustment has been made with sufficient accuracy by measuring the ratio and by testing against standards respectively. (3) The disturbances from unequal temperature may be reduced by observing from a distance if necessary, and by securing the balance from other sources of radiation. It is difficult to determine when the disturbance from this source is sufficiently removed. (4) The buoyancy may be allowed for by calculating the difference of weight of air displaced by the weights and object in the usual manner. This will require an approximate knowledge of the specific gravity or of the volume of the object, and measurements of the temperature, pressure, and possibly humidity also, of the air in the balance case at the time of weighing. The object must of course not be losing weight by evaporation or otherwise, or gaining by condensation. On no account must the object be at a temperature considerably different from that of

the air in the balance case or the latter be materially different from that of the air of the room.

These four sources of error must be reduced to be of negligible effect compared with the deviation measure of the result as already stated. By the assumption of $n = 25$ as given at page 29, and from the value *a.d.* = 0.00 021 found in the preliminary trial, we find $A.D. = \frac{0.00\,021}{\sqrt{25}} = 0.00\,004\,2$. This is probably beyond the limit actually attainable on the balance reading to only 0.1 mgr. direct, unless the method of weighing by swings is used. Moreover, ordinarily the time occupied in making 25 independent weighings would be prohibitive. It would probably give more nearly the practicable limit to use $n = 4$ to 9, i.e. $A.D.$ = 0.00 010 to 0.00 007 0. We will then use $A.D.$ = 0.00 007 grms. as an approximate value of the *d.m.* To be negligible then the residual from each of the four sources of error must be equal to or less than $\frac{1}{3}A.D./\sqrt{p}$ where $p = 4$. The limit is therefore $\frac{1}{3} \times 0.00\,007/2 = 0.00\,001\,2$ or nearly enough 0.00 001 0 grms. = 0.01 mgr.

(1) Therefore to make the error from the ratio of the balance arms negligible this ratio must be adjusted to a corresponding amount, or the ratio must be measured for a correction with that degree of accuracy. The ratio of the balance arms enters as a direct factor and is very nearly unity. An error of 0.00 001 grms. in 34 would be 0.00 000 03. The ratio must then be determined to this fraction, i.e. to 3 in 10 000 000. It would probably be impracticable to adjust the arms as close as this limit. But an inspection of the method and formula for measuring the ratio shows that this limit could perhaps be reached in that measurement by making several observations.

(2) The limit of 0.01 mgr. in the adjustment of the weights, or in the comparison of them with a standard, cannot be reached with any means ordinarily at hand, if at all. The best that can be done without undue labor is probably to get the errors of the weights within about 0.1 mgr. This would be a residual of about the same magnitude as the *d.m.* of the weighing.

(3) The disturbances due to unequal temperature and air currents are not easily estimated. They would be rendered as small as practicable by using the balance in a room of nearly constant temperature, and by screening the balance from any objects having high or low temperatures and from draughts of air. It would be advantageous to have the final readings taken by the observer at a distance, using a telescope and the method of swings. The amount of the residual could probably not be determined. It is doubtful whether it could be reduced to the limit 0.01. But as the disturbances would probably not be of the same sign and amount at different times, they would appear as a part of the final *d.m.* The weighing should, of course, be taken on different days and at different hours in the same day. We will assume this residual to be negligible.

(4) For the buoyancy correction we will suppose the air in the balance case to be dry. Let the specific gravity of the substance be about 2, that of the weights being 8.5. Then the volume of air displaced by the substance would be $34/2 = 17$ cc., that by the weights $34/8.5 = 4$ cc. The substance would therefore appear too light by the weight of $17 - 4 = 13$ cc. of air at the density of that in the balance case at the time. Suppose the observed temperature and pressure of the air to be 16° C. and 760 mm. The weight of 1 cc. of dry air under these conditions is 1.2 mgr., and that of 13 cc. is 16 mgr. This must be known to the limit 0.01 mgr. in order that the residual be negligible. The density changes by 0.004 mgr. per cc. for 1° in temperature, so that for 13 cc. the change would be 0.05 mgr. The limit would then correspond to an error of $\frac{1}{5} = 0°.2$ C. The change of density of air per mm. change of pressure at 760 mm. is 0.0015 mgr. per cc. This for 13 cc. is 0.02 mgr., or twice the limit. As both temperature and pressure are variable together the limit must be $0.01/\sqrt{2}$; so that the accuracy necessary would be respectively $0.2/\sqrt{2} = 0°.14$ and $0.5 \text{ mm.}/\sqrt{2} = 0.35$ mm. These limits could be reached only with great care. The question would then remain as to whether the weight of a cc. of dry air was known with suf-

ficient accuracy. The limit required is 0.01 mgr. in 0.016 grms., or about 0.05 per cent. It is doubtful if the values for the density of air given in the tables can be relied on as applying to ordinary air with this closeness, especially when the humidity of the air is neglected.

It appears then that the residuals from (1) and (4) may be rendered nearly but not quite negligible, that the residual from (3) cannot be well determined, but is assumed to be negligible, and so far as it exists will appear as a part of the *d.m.*, and that the residual of (2) will be about equal to the *d.m.* Hence the precision measure of the result will be about, but probably somewhat greater than

$$\delta = \sqrt{(0.07^2 + 0.10^2)} = 0.12 \text{ mgr. approx.}$$

There will be no constant errors large with respect to this, so far as we can discover, so that the estimate of the accuracy of the result will be about 0.12 mgr.

To summarize we should say that to obtain the best result practicable by the given balance, the following is necessary:

Ratio of arms must be determined to 6 in 10 000 000.

Weights should be corrected to at least 0.1 mgr.

Screening from radiation and draughts should be thorough.

Temperature of balance case must be measured to 0°.14 C., and air must be dry.

Reduced barometric pressure at time must be found to 0.35 mm.

Constant for weight of air must be known well within 0.1 per cent.

The result will then be accurate to about 0.1 mgr.

Example IV.—*Problem.* Desired the measurement of a voltage x of about 110 volts with an accuracy of 0^v.2, using a Weston magnetic voltmeter which is graduated to single volts and read to 0^v.1 by estimation, the conditions being such that only a single observation can be taken when x is being read. Resistance of voltmeter about 17 000 ohms.

Solution.—To reach this limit we must be able to get a *p.m.* of less than 0v.2 when all determinate errors are eliminated.

We must first find the deviation measure. Suppose that several measurements made on a constant voltage of 110v. showed an *a.d.* of 0v.06. Lacking this test we should probably assume about this amount as the *a.d.*, for the following reasons. The deviation would be due to errors of estimation almost wholly. If the index were fine enough, the *a.d.* of estimating the tenths would be 0v.025 (Special Law of Deviations I, page 21), but with the usual size of index, of deviations, and the unavoidable parallax 0v.05 to 0v.1 would be a safer assumption.

The discoverable sources of instrumental error may be classified as

(1) Changes due to change of temperature;

(2) Permanent alterations of resistance;

(3) Accidental irregularities in spacing the graduations.

(4) Graduation not being correct volts, whether owing to faulty graduation at outset or to change of strength of magnets.

Of these, (1) and (3) cannot well be eliminated, but (2) and (4) can be determined, (e.g. by comparison with a Clark cell) at every 10 volts, more or less, along the scale and corrections applied. The points whose errors are thus found will be called the calibrated points, and the corrections, the calibration corrections.

The precision measure δ being 0v.2 we have

$$\delta^2 = d.m.^2 + r_1^2 + r_2^2 + r_3^2 + r_4^2,$$

and to put in the work to the best advantage we shall make $r_1^2 = r_2^2 = r_3^2 = r_4^2$ approx. Thus

$$0.2^2 = 0.06^2 + r_1^2 + r_2^2 + r_3^2 + r_4^2,$$

$$\therefore r_1^2 = r_2^2 = r_3^2 = r_4^2 = \frac{0.2^2 - 0.06^2}{4} = \frac{0.036}{4} = 0^v.009;$$

$$\therefore r = 0^v.09 \text{ approx.},$$

which is therefore the normal limit for the residuals.

(1) By the statement of the makers, the temperature error for the magnetic voltmeter is 0.01 per cent. per degree centigrade. At 110^{v} the limit 0^{v}.09 is 0.09/110 = 0.08 per cent approx. This corresponds to a change of 8° C. in the temperature of the whole instrument; for this correction is not for the heating of the coils by the current alone, but is understood to refer to a change of the whole instrument. Apart from the effect due to greater heating of the coil than of the remaining parts, this error might be made negligible by observing the temperature and correcting. This, however, is not practicable and would probably be of doubtful value.

(2) An accidental change of resistance of the coils would cause an error proportioned to the change. An error of the limiting amount, that is of 0.08 per cent, would be produced then by an accidental change of 0.00 08 × 17 000 = 14$^{\omega}$. A change of half this amount can be easily detected by measurements on a bridge from time to time, and therefore this source of error can be made negligible.

(3) These irregularities must not exceed an average of 0^{v}.09 or practically 0.1 divisions between the calibrated points. Their amount, however, is practically not determinable, and their effect can be removed only by calibration at many points.

(4) The errors from this source must be corrected by calibration with Clark cell or otherwise, and with a residual not exceeding 0^{v}.09. The Clark cell method is an indirect measurement, and would therefore be discussed separately by the methods later given, and will merely be summarized here. The expression for the voltage at the terminals of the voltmeter by one method is

$$V = E[1 - a(t - 15)]\frac{R}{r},$$

where E = E.M.F. of cell at 15° C., a = temp. coeff. = 0.00 038 for Carhart-Clark cell, t = temp. of cell, R = res. of voltmeter, r = res. between terminals of cell. The results of such a discussion show that, for a result accuracy of 0^{v}.1 in V, the resid-

uals for the various component quantities must be as follows: For E, 0^v.00 056 (just attainable); for a, 0.00 008 (easily made negligible); for t, 1°.0 (easily reduced to 0°.5, and made negligible); for R, 0.04 per cent (attainable); for r, 0.04 per cent (attainable). As indicated by the comments in the parentheses, we can do perhaps a little better than 0^v.1 in V. But the calibration requires also a reading of the voltmeter when the voltage is V at its terminals, and the *a.d.* of this reading is 0^v.06. Hence several readings must be taken at each point to be calibrated. If this is done we may expect to get the calibration error with a residual not much exceeding 0^v.1 or 0^v.15.

Taking the *d.m.* and all the residuals into consideration we may then hope to get an accuracy as follows:

$$\delta^2 = 0.06^2 + 0.1^2 + 0.1^2 + 0.0 + 0^v.15 = 0.046.$$

$$\therefore \delta = 0^v.21;$$

that is, of the prescribed amount. Evidently this limit cannot be reached, however, without great care and the best instruments, and it is not to be assumed without experimental proof that the instrument will remain long without a change exceeding this amount.

INDIRECT MEASUREMENTS.

Estimate of Accuracy of Indirect Result.—The terms *accuracy* and *precision* are used with the same significance in connection with indirect as with direct measurements.

When the estimated accuracy of an indirect result is stated to be of a certain amount it is thereby meant that the precision measure of the result is of that amount, and that, so far as can be discovered, there is no constant error in the result which is sensible (i.e., not negligible) compared with this precision measure.

When it is stated that a result is desired with a specified accuracy, it is similarly meant that the precision measure must not exceed that amount, and that no discoverable constant error of an amount not negligible in comparison with this must be left in the result.

Thus by stating the accuracy we do not mean that the actual error of the result is of just that amount, for if we did we should correct accordingly. Neither do we mean that this is a more probable value of the error than any other. But using the average deviation as the deviation measure, we mean that, so far as we can discover, the average effect of all the errors remaining is of the stated amount, and may be either positive or negative. This implies that if several results were to be obtained under the same conditions by the same

method, apparatus, etc., the average discrepancy amongst them would be approximately of this amount.

Thus to be able to estimate the accuracy of an indirect result, we must have data for finding properly the precision measures of its component direct measurements, and we must be able to compute the precision measure of the result from that of the components. Also we must be able to show that due care has been taken in the correction or removal of all the determinate errors of the components, so that none but negligible constant errors remain. Finally, we must be able to show that the "error of method" (see next paragraph) is negligible, or if not so to take it into account.

The numerical part of the estimate of accuracy will be the *P.M.* if the "error of method" is negligible, or will be the $\sqrt{(P.M.^2 + R^2)}$ if R be the estimated amount of this error.

Error of Method.—Besides the constant error of the components there is, in certain cases, another source of error in indirect results. The method adopted may have inherent sources of error; that is, it may fail to give correct results, however accurate the components, because the result sought does not bear the supposed relation to the components, as expressed by the function from which the result is computed. This may arise through the introduction of some approximation, through the existence of some inexact hypothesis, or from other similar cause. Errors of this sort will be called *errors of method.* Their amount or the average uncertainty arising from them may sometimes be estimated and allowed for. The possibility of their existence should not be overlooked. As an instance, we may cite the case of the "Stray Power Method" of testing dynamos or motors (explained among the final illustrative problems). In this method the assumption is made that the "stray power" is the same when the machine is running under no load as when under full or partial load, certain conditions being fulfilled. The results by this method will be more or less uncertain if this assumption is not strictly true.

Check methods are of course of the same importance in detecting and eliminating constant errors of indirect as of direct measurements. To obtain the most accurate results and to get clues to constant errors independent results should be obtained by as many different methods as possible. Such checks may be had by changing the methods or instruments for measuring the components, or by changing the indirect process for another. As an instance of the latter way we may take the determination of the commercial efficiency of an electric motor. Several checks on this might be obtained by using the various purely electrical methods, and still other checks by testing on a cradle dynamometer, with a friction brake, etc.

Relation between the Precision Measure of Result and Components.—*Types of Problems.* The following three types of problems are the chief ones which arise concerning the relation between the precision measure, *P.M.*, of the result and those, *p.m.*, of its components.

First. *To find the P.M. of the result given the p.m. of each component.*

Second. *To find the best value for the p.m. of each component, for a prescribed P.M. of the result.*

Third. *Given the P.M. of the result, or the p.m. of some of the components, or both, to find the best magnitudes* m_1, m_2, *etc., of the components.*

The third of these, when solvable, enables us to determine in advance the best relative or actual magnitudes of the components, that is those which will yield the most precise result with the given apparatus. Examples of this occur in such cases as finding the best deflection to employ in using an ordinary tangent galvanometer, the best deflections to use in finding battery E.M.F. or resistance by the two-deflection method, etc. These problems will be again taken up after the first two types have been entirely discussed.

The next step will therefore be to establish formulæ for the relation between the *P.M.* of the result and the *p.m.* of the components for the solution of the first two types of problems.

General Formulæ. Let $x, y, z, \ldots$ represent a number of *independent* variables. Then if w is a quantity which is some function f of these, this fact is ordinarily indicated by an expression of the form

$$w = f(x, y, z, \ldots). \quad . \quad . \quad . \quad . \quad . \quad . \quad [16]$$

This expression, and others deducible from it, apply immediately to indirect measurements and their precision discussion.

For example, if g were determined by means of a simple pendulum, the process would be an indirect measurement of g. The component quantities directly measured would be l, the length, and t, the time of a single vibration of the pendulum; and the function by which g is deduced from these would be

$$g = \pi^2 \frac{l}{t^2}.$$

This corresponds to the above general expression thus: g to w, l to x, and t to y; for l and t are independent quantities, and π is a constant.

Instead of the ordinary mathematical notation using x, y, and z as in equation (**16**), it will be much more convenient for this work to employ a special one adapted to the purpose. We will then use as the general expression

$$M = f(m_1, m_2, \ldots, m_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad [17]$$

where M is any quantity whatever which is a function of the quantities $m_1, m_2, \ldots, m_n$, each of which is directly measured and is wholly independent of the others.

A quantity M thus determined will here be called an indirect or an indirectly measured quantity. It is often called a derived quantity.

Notation.—The following notation will be always employed :—

M = the final indirect result.
m = any component.
a, b = constants.
n = the number of directly measured components.
Δ = any small finite change (expressed in units) in M.
δ = " " " " " " " " m.

$\frac{\Delta}{M}$ = the corresponding fractional change in the result M.

$\frac{\delta}{m}$ = " " " " " any m.

ΔM and δm will be used instead of Δ and δ when needed for greater clearness.

Any change will be considered small which does not exceed a few per cent of the corresponding quantity.

$f( )$ will be written for brevity in place of $f(m_1 m_2, \ldots m_n)$.

Corresponding subscripts will denote corresponding quantities. Thus Δ_1 will denote a change in M corresponding to a change δ_1 in m_1, etc., and *vice versa*.

The subscript $_k$ will be used to denote a general term. Thus m_k will denote *any* component, and δ_k or Δ_k the corresponding changes in M.

Separate Effects.—I. What will be the change Δ_1 in M corresponding to, or produced by, a change δ_1 in m_1, all other components remaining unchanged?

Differentiating $f( )$ with respect to m_1, all other components being considered constant, we have as the rate of change of M with m_1

$$\frac{dM}{dm_1} = \frac{df( )}{dm_1}.$$

Passing from the limit to finite changes, we have

$$\frac{\Delta_1}{\delta_1} = \frac{dM}{dm_1} \text{ approx.},$$

the approximation being closer as the changes Δ_1 and δ_1 are smaller. Hence

$$\frac{\Delta_1}{\delta_1} = \frac{df()}{dm_1}. \quad \ldots\ldots\ldots \quad [18]$$

$$\therefore \ \Delta_1 = \frac{df()}{dm_1} \cdot \delta_1. \quad \ldots\ldots\ldots \quad [19]$$

Similarly for a change δ_2 in m_2, the corresponding change in M will be

$$\Delta_2 = \frac{df()}{dm_2} \cdot \delta_2, \quad \ldots\ldots\ldots \quad [20]$$

and so on for all the n components.

Example XIII, page 86.

II. Given a small change Δ in M, what change δ in any component m alone would correspond to, or would be necessary to produce, this change Δ? This is evidently merely the converse of I.

Consider first m_1. By 18 we have

$$\frac{\Delta_1}{\delta_1} = \frac{df()}{dm_1},$$

$$\therefore \ \delta_1 = \Delta_1 \Big/ \frac{df()}{dm_1}. \quad \ldots\ldots\ldots \quad [21]$$

Similarly,

$$\delta_2 = \Delta_2 \Big/ \frac{df()}{dm_2}, \quad \ldots\ldots\ldots \quad [22]$$

and so on for all the n components.

The changes Δ_1, Δ_2, etc., may or may not be equal.

Example XIV, page 86.

Resultant Effects.—III. Suppose that small changes δ_1 in m_1, δ_2 in m_2, and so on, occur simultaneously, what will be the resulting change in M? Here we must distinguish two cases.

1°. Where the changes δ_1, δ_2, etc., are of specified magnitude and sign.

2°. Where we wish to deduce a general expression which will give us the best solution when all that we know respecting the δ_1, δ_2, etc., is the following. Each δ may be of either sign, + or —, and of a magnitude following a certain law of distribution, i.e. the δ's follow the "general law of deviations." It is this second case which gives us the formulæ to be used in precision discussions.

In either case let Δ_1, Δ_2, etc., denote the changes in the result M corresponding respectively to the separate changes δ_1, δ_2, etc., in the components m_1, m_2, etc., as in formulæ 19 and 20.

1°. For the first case we know that, as m_1, m_2, etc., are independent, the resultant change Δ in M due to the simultaneous occurrence of the changes Δ_1, Δ_2, etc., will be the algebraic sum of these separate changes, as may be denoted by

$$\Delta = \Delta_1 + \Delta_2 + \ldots \Delta_n. \quad . \quad . \quad . \quad . \quad [23]$$

This expression is, however, almost never made use of in the following work, as the conditions for which it holds rarely occur.

2°. For the second case, the desired expression may be reached as follows.

If the changes Δ_1, Δ_2, etc., going to make up Δ follow the general law of deviations, then by the Method of Least Squares it is shown that the best or most probable value of Δ will be given by

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad [24]$$

By "most probable value" it is meant that the value would have greater probability than one obtained by any other method; or in other words, if we had the means of putting it to proof we should find, in an indefinitely long experience, that the value of Δ given by the formula would be nearer the truth than one obtained by any other expression. This value of Δ

is the best in the same sense that the arithmetical mean is the best representative value of a series of observations.

Two other points may be demonstrated from a consideration of the general law of deviations. *First.* If $\Delta_1, \Delta_2, \ldots \Delta_n$ follow this law, then Δ will follow the same law. *Second.* If $\Delta_1 = \frac{df()}{dm_1} \cdot \delta_1$, etc., have the significance given to them in I, then if δ_1 have various values following the general law of deviations, Δ_1 will have corresponding values following the same law. The first of these may be understood from the consideration that obviously in 24 if Δ_1^2, Δ_2^2, etc., each and all follow the same law of distribution, then Δ^2 must follow the same law; and that the general law is given by an exponential expression such that its form is not changed by squaring. The second is obviously true, because for a given function $\frac{df()}{dm_1}$ is constant, and thus Δ_1 is merely δ_1 multiplied by a constant, so that both must follow the same law of distribution.

Substituting therefore for Δ_1, etc., their expressions in terms of δ_1, etc., given by 19 and 20, we have

$$\Delta^2 = \left(\frac{df()}{dm_1} \cdot \delta_1\right)^2 + \left(\frac{df()}{dm_2} \cdot \delta_2\right) + \ldots + \left(\frac{df()}{dm_n} \cdot \delta_n\right)^2. \quad . \quad [25]$$

This expression then gives us the most probable value of the change Δ which is the resultant effect of, or corresponds to, the simultaneous changes δ_1, δ_2, etc., in the components, all of these changes following the general law of distribution of deviations.

These two expressions, 24 and 25, are the fundamental ones which underlie all the following work. The solution given by the expression 24 is not an exact one. The method and conditions under which it is deduced are such that we know that it gives merely the most probable value of Δ, that is, it gives in the long-run better values of Δ than any other expression gives for the same conditions respecting the δ's. We know further that the value of Δ obtained by it is entitled to greater weight

the larger the number n of the components, $m_1, m_2, \ldots m_n$; the more closely the δ's follow any one of the laws of distribution of deviations; and the smaller the value of Δ as compared with M.

Example XV, page 87.

If in 24 we divide both sides by M, we have

$$\frac{\Delta^2}{M} = \frac{\Delta_1^2}{M} + \frac{\Delta_2^2}{M} + \ldots + \frac{\Delta_n^2}{M}, \quad \ldots \quad [26]$$

an expression to which reference will occasionally be made.

IV. If we were to ask for the resultant effect the question analogous to II for separate effects, it would be, what simultaneous changes in $\delta_1, \delta_2, \ldots, \delta_n$ would produce a specified resultant change Δ in M? It is obvious that we might have an infinite number of values of δ_1, δ_2, etc., which would produce the specified Δ for any given function $f()$. It is therefore necessary to assign some further condition.

Equal Effects.—For reasons which will appear later, we may advantageously restrict our inquiry to the case where each δ produces an *equal effect* with every other on the value of Δ, i.e., where each δ produces in M the same change as every other. This will evidently be the case when in 24 we have

$$\Delta_1 = \Delta_2 = \ldots = \Delta_n, \quad \ldots \quad [27]$$

as Δ_1 is the change in M due to δ_1, Δ_2 to δ_2, and so on.

For the general case, then, where the δ's follow the law of deviations (corresponding to 24 and 25) we have the following: For a specified value of Δ, the values of $\delta_1, \delta_2, \ldots, \delta_n$ which will produce this Δ under the condition of equal effects are such that

$$\Delta_1 = \Delta_2 = \ldots = \Delta_n = \frac{\Delta}{\sqrt{n}}, \quad \ldots \quad [28]$$

where n is the number of components, m, entering into the given function.

Substituting the expressions for δ_1, δ_2, etc., in terms of Δ_1, Δ_2, etc., we have further

$$\delta_1 = \frac{\Delta}{\sqrt{n}} \Big/ \frac{df()}{dm_1}, \quad \ldots\ldots\ldots \quad [29]$$

$$\delta_2 = \frac{\Delta}{\sqrt{n}} \Big/ \frac{df()}{dm_2}, \quad \ldots\ldots\ldots \quad [30]$$

. . . . .

$$\delta_n = \frac{\Delta}{\sqrt{n}} \Big/ \frac{df()}{dm_n}. \quad \ldots\ldots\ldots \quad [31]$$

Example XVI, page 88.

These are also fundamental expressions, and it will be seen by inspection of the method of their deduction that the values of δ arrived at by them form merely a special set of values out of the infinite number of possible values which would satisfy the simple condition that the δ's must follow the law of deviations.

Applications to Precision Discussions.—In the application of these formulas to indirect measurements, we shall let δ_1, δ_2, etc., represent the precision measures *p.m.* of the components. Then Δ, being, as shown, necessarily a quantity of the same kind as δ, will be the precision measure *P.M.* of the indirect result.

That this application of the formulas is justifiable may be shown as follows.

The only condition imposed in the deduction of the fundamental expression $\Delta^2 = \Delta_1^2 + etc.$, is that the quantities Δ_1, Δ_2, *etc.*, follow the general law of deviations. But as shown at page 52, Δ_1, Δ_2, *etc.*, will follow the law of deviations if the values of δ_1, δ_2, etc., do so. Now the *p.m.* which is used as δ is a quantity of the same nature as the deviation measure, being calculated by combining the *d.m.* with the residuals, which are also deviation measures. And the deviation measure follows the law of deviations, being merely special value of the deviations. Thus the *p.m.* and therefore the values of

δ_1, δ_2, *etc.*, and consequently of Δ_1, Δ_2, *etc.*, follow that law. Therefore the general expression is applicable when we use precision measures for the values of δ.

The same form of statement would be true whatever kind of precision measure were employed, but it is obviously essential that in a given case the δ's must all be of the same kind. For instance, if the average deviation is employed for one component, it must be for all in the same computation. But we must note on the other hand that if, for example, m_1 in a given measurement is a quantity observed but once, while m_2 is a mean of several observations, the deviation measure δ_1 of m_1 must of course be deduced from the average deviation (*a.d.*) of a single observation, and the precision measure δ_2 of m_2 must be deduced from the *A.D.*, the average deviation of the mean. For the *a.d.* is a deviation measure of the same kind and order relatively to the single observation used as m_1, as the *A.D.* is of the mean result employed for m_2. The resulting value of M must be regarded as a single observation, and its Δ must be considered as a *a.d.* if any one or more of the components is a single measurement. But if all of the components used are mean results, the M would be considered a mean, and the Δ a *A.D.*

Formulæ for General and Special Functions $f(\ )$.— There are several special forms of the function $f(\ )$ for which it is advantageous to have formulæ giving Δ in terms of δ, or the converse, and formulæ for equal effects. For completeness the general expressions are first given and then those for the special functions.

$$M = f(m_1, m_2, \ldots, m_n). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad [32]$$

Separate Effects,—

$$\Delta_1 = \frac{df(\ )}{dm_1}.\delta_1, \qquad \Delta_2 = \frac{df(\ )}{dm_2}.\delta_2, \ldots \text{ etc.} \quad . \quad . \quad [33]$$

Conversely

$$\delta_1 = \Delta_1 \Big/ \frac{df(\ )}{dm_1}, \qquad \delta_2 = \Delta_2 \Big/ \frac{df(\ )}{dm_2}, \ldots \text{etc.} \quad . \quad [34]$$

Resultant Effect,—

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2. \quad . \quad . \quad . \quad . \quad [35]$$

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\Delta_1}{M}\right)^2 + \left(\frac{\Delta_2}{M}\right)^2 + \ldots + \left(\frac{\Delta_n}{M}\right)^2. \quad . \quad . \quad [36]$$

Equal Effects,—

$$\Delta_1 = \Delta_2 = \ldots = \Delta_n = \frac{\Delta}{\sqrt{n}}. \quad . \quad . \quad . \quad . \quad [37]$$

$$\frac{\Delta_1}{M} = \frac{\Delta_2}{M} = \ldots = \frac{\Delta_n}{M} = \frac{1}{\sqrt{n}} \frac{\Delta}{M}. \quad . \quad . \quad . \quad [38]$$

Examples XIII–XVI illustrate the application of these formulæ to precision discussions if the term *precision* or *precision measure* be substituted for "change."

$$M = \pm m_1 \pm m_2 \pm \ldots \pm m_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [39]$$

Separate Effects,—

$$\Delta_1 = \pm \delta_1, \qquad \Delta_2 = \pm \delta_2, \ldots \qquad \Delta_n = \pm \delta_n. \quad . \quad . \quad [40]$$

Conversely

$$\delta_1 = \pm \Delta_1, \qquad \delta_2 = \pm \Delta_2, \ldots, \qquad \delta_n = \pm \Delta_n. \quad . \quad [41]$$

Resultant Effect,—

$$\Delta^2 = \delta_1^2 + \delta_2^2 + \ldots + \delta_n^2. \quad . \quad . \quad . \quad . \quad [42]$$

Equal Effects,—

$$\delta_1 = \delta_2 = \ldots = \delta_n = \frac{\Delta}{\sqrt{n}}. \quad . \quad . \quad . \quad . \quad [43]$$

Example XVII, page 88.

For separate effects the formula may be stated in words as follows. For a sum or difference the change in the result is numerically equal to the change in the component, and of the same sign.

Deduction. $\frac{df(\ )}{dm_1} = \pm \delta_1$, etc. Whence by substitution in the general formulæ we have the above results.

$$M = am_1 + bm_2 + \ldots + km_n. \quad \ldots\ldots\ldots\ldots\ldots \quad [44]$$

$a, b, \ldots, k$ = constants, and may be either + or − or of indeterminate sign and of any magnitude. This case therefore includes the preceding.

Separate Effects,—

$$\Delta_1 = a\delta_1, \quad \Delta_2 = b\delta_2, \quad \ldots, \quad \Delta_n = k\delta_n. \quad \ldots \quad [45]$$

Conversely

$$\delta_1 = \frac{1}{a}\Delta_1, \quad \delta_2 = \frac{1}{b}\Delta_2, \quad \ldots, \quad \delta_n = \frac{1}{k}\Delta_n. \quad \ldots \quad [46]$$

Resultant Effect,—

$$\Delta^2 = (a\delta_1)^2 + (b\delta_2)^2 + \ldots + (k\delta_n)^2. \quad \ldots \quad [47]$$

Equal Effects,—

$$a\delta_1 = b\delta_2 = \ldots = k\delta_n = \frac{\Delta}{\sqrt{n}}. \quad \ldots \quad [48]$$

Deduction. $\frac{df(\ )}{dm_1} = a$, etc., and substituting these in the general formula gives the above results.

$$M = a \cdot m_1 \cdot m_2 \cdot \ldots \cdot m_n \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad [49]$$

a = constant factor.

Separate Effects,—

$$\frac{\Delta_1}{M} = \frac{\delta_1}{m_1}, \quad \frac{\Delta_2}{M} = \frac{\delta_2}{m_2}, \quad \ldots, \quad \frac{\Delta_n}{M} = \frac{\delta_n}{m_n}. \quad \ldots \quad [50]$$

Conversely

$$\frac{\delta_1}{m_1} = \frac{\Delta_1}{M}, \quad \frac{\delta_2}{m_2} = \frac{\Delta_2}{M}, \quad \ldots, \quad \frac{\delta_n}{m_n} = \frac{\Delta_n}{M}. \quad \ldots \quad [51]$$

Resultant Effect,—

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\delta_1}{m_1}\right)^2 + \left(\frac{\delta_2}{m_2}\right)^2 + \ldots + \left(\frac{\delta_n}{m_n}\right)^2. \quad \ldots \quad [52]$$

Equal Effects,—

$$\frac{\delta_1}{m_1} = \frac{\delta_2}{m_2} = \ldots = \frac{\delta_n}{m_n} = \frac{1}{\sqrt{n}} \frac{\Delta}{M}. \quad \ldots \quad [53]$$

Example XVIII, page 89.

For separate effects the result may be put into words by saying that for any factor the fractional change in the result is equal to the fractional change in the factor.

Deduction. $\frac{df()}{d.m_1} = a \cdot m_2 \cdot \ldots \cdot m_n = \frac{M}{m}$, etc., and substituting in 33 gives $\Delta_1 = \frac{M}{m_1} \cdot \delta_1$, $\therefore \frac{\Delta_1}{M} = \frac{\delta_1}{m_1}$. Similarly, $\frac{\Delta_2}{M} = \frac{\delta_2}{m_2}$, etc., for separate effects. Substituting these in the general formulæ 36 and 38 gives 52 and 53.

$$M = \frac{a \cdot m_1 \cdot m_3 \ldots}{b \cdot m_2 \cdot m_4 \ldots} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad [54]$$

Separate Effects,—

$$\frac{\Delta_1}{M} = \frac{\delta_1}{m_1}, \quad \frac{\Delta_3}{M} = \frac{\delta_3}{m_3}, \ldots \quad \ldots \quad [55]$$

$$\frac{\Delta_2}{M} = -\frac{\delta_2}{m_2}, \quad \frac{\Delta_4}{M} = -\frac{\delta_4}{m_4}, \ldots \quad \ldots \quad [56]$$

Converse is evident.

Resultant Effect,—

Same as 52 since negative signs disappear on squaring. [57]

Equal Effects,—

$$\frac{\delta_1}{m_1} = -\frac{\delta_2}{m_2} = \frac{\delta_3}{m_3} = -\frac{\delta_4}{m_4} = \ldots = \frac{1}{\sqrt{n}}\frac{\Delta}{M}. \quad [58]$$

N.B.—The signs are of no importance and may be neglected in most precision discussions, for they merely indicate whether a + change in the result is caused by a + or a — change in the component, a fact usually of no interest.

Deduction.

$$\frac{df()}{dm_1} = \frac{M}{m_1}, \text{ as before}; \quad \frac{df()}{dm_2} = -\frac{a \cdot m_1 \cdot m_3 \ldots}{b \cdot m_2^2 \cdot m_4 \ldots} = -\frac{M}{m_2};$$

and so on. Substituting in general formulæ gives above results as before.

$$M = a \cdot m^v. \quad [59]$$

a = constant factor; v = constant exponent.

Separate Effect,—

$$\frac{\Delta}{M} = v\frac{\delta}{m}. \quad [60]$$

In words: If the function be a constant power, and he either with or without a constant coefficient, the fractional change in the result is equal to the fractional change in the component multiplied by the exponent of the power.

Conversely

$$\frac{\delta}{m} = \frac{1}{v}\frac{\Delta}{M}. \quad [61]$$

Example XIX, page 90.

N.B.—As the value of v is unrestricted, this holds for a negative exponent. Thus if $v = -c$,

$$M = am^{-c} = \frac{a}{m^c}, \text{ and } \frac{\Delta}{M} = -c\frac{\delta}{m}. \quad . \quad . \quad [62]$$

Deduction.

$$\frac{df(\,)}{dm} = avm^{v-1} = v\frac{M}{m} \quad \therefore \Delta = v\frac{M}{m}\delta \text{ and } \frac{\Delta}{M} = v\frac{\delta}{m}.$$

$$M = a \cdot m_1^{v} \cdot \ldots \cdot m_n^{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [63]$$

a = constant factor; v, w = constant exponents.

Separate Effects,—

$$\frac{\Delta_1}{M} = v\frac{\delta_1}{m_1}, \quad \ldots, \quad \frac{\Delta_n}{M} = w\frac{\delta_n}{m_n}. \quad . \quad . \quad . \quad [64]$$

Conversely

$$\frac{\delta_1}{m_1} = \frac{1}{v}\cdot\frac{\Delta_1}{M}, \quad \ldots, \quad \frac{\delta_n}{m_n} = \frac{1}{w}\frac{\Delta_n}{M}. \quad . \quad . \quad . \quad [65]$$

Resultant Effect,—

$$\left(\frac{\Delta}{M}\right)^2 = \left(v\frac{\delta_1}{m_1}\right)^2 + \ldots + \left(w\frac{\delta_n}{m_n}\right)^2. \quad . \quad . \quad . \quad . \quad [66]$$

Equal Effects,—

$$v\frac{\delta_1}{m_1} = \ldots = w\frac{\delta_n}{m_n} = \frac{1}{\sqrt{n}}\frac{\Delta}{M}. \quad . \quad . \quad . \quad . \quad . \quad [67]$$

Example XX, page 91.

N.B.—As v, w, etc., are unrestricted, any of the exponents may be negative, so that this covers the case where any of the factors are in the denominator. The only difference will be that in separate or equal effects the sign of the exponent will become negative, as shown in formula [62] above. But as

the sign is usually of no interest as already explained, we may omit the consideration of it, and in using this formula for factors we may treat a factor in the denominator, i.e., with a negative exponent, precisely as if it were in the numerator, i.e., had a positive exponent.

This case evidently includes as special cases all the foregoing functions separable into factors.

Demonstration.—Obvious from the two preceding.

Besides the foregoing simple functions which consist merely of the sum or difference, product, quotient or power of the direct quantities, there are others less simple, for which formulæ for equal effects are of much service. Of these the three principal ones are, first, where $f()$ can be separated into a series of factors each of which is a function of one component direct measurement and only one; second, where $f()$ can be separated into two or more terms to be added or subtracted, each of which is a function of several of the components but has no components common to any two of the functions; and third, where $f()$ can be separated into two or more factors each of which is a function of several of the components and having no component common to any two of the functions. In the second and third case each function comprises a group of components, and the process will hence be referred to as *separation into groups*. Of course any of these functions $f()$ could be discussed by the general formulæ, the advantage derived from the use of the special ones being that they greatly lessen the work of differentiation and numerical substitution, where $f()$ is complicated, as is often the case; and they also render the solution of the problem clearer and easier to follow.

Separation into Factors which are Functions of Single Components.

$$M = \phi(m_1) \cdot \rho(m_2) \cdot \ldots \cdot \sigma(m_n) \quad [68]$$

Resultant Effect,—

$$\left(\frac{\Delta}{M}\right)^2 = \left[\frac{\delta\phi(m_1)}{\phi(m_1)}\right]^2 + \left[\frac{\delta\rho(m_2)}{\rho(m_2)}\right]^2 + \ldots + \left[\frac{\delta\sigma(m_n)}{\sigma(m_n)}\right]^2. \quad [69]$$

Equal Effects,—

$$\frac{\delta\phi(m_1)}{\phi(m_1)} = \frac{\delta\rho(m_2)}{\rho(m_2)} = \ldots = \frac{\delta\sigma(m_n)}{\sigma(m_n)} = \frac{1}{\sqrt{n}}\frac{\Delta}{M}. \quad . \quad [70]$$

These determine the values of $\frac{\delta\phi(m_1)}{\phi(m_1)}$, etc., from the stated value of $\frac{\Delta}{M}$, and from these values we have to find the corresponding values of $\frac{\delta}{m}$ or of δ, as the case may require, by the general or special formulæ for separate effects.

Example XXI, page 91.

Deduction.—$\Delta_1 = \frac{df(\,)}{dm_1}\cdot\delta_1$. Now

$$\frac{df(\,)}{dm_1} = [\rho(m_2)\cdot\ldots\cdot\sigma(m_n)]\frac{d\phi(m_1)}{dm_1} = \frac{M}{\phi(m_1)}\cdot\frac{d\phi(m_1)}{dm_1}.$$

$$\therefore \frac{\Delta_1}{M} = \frac{1}{\phi(m_1)}\cdot\frac{d\phi(m_1)}{dm_1}\cdot\delta_1 = \frac{\delta\phi(m_1)}{\phi(m_1)}\cdot\frac{\delta_1}{\delta_1} \text{ approx.} = \frac{\delta\phi(m_1)}{\phi(m_1)}.$$

Similarly,

$$\frac{\Delta_2}{M} = \frac{\delta\rho(m_2)}{\rho(m_2)}, \text{ etc.}$$

By [26] for resultant effect we must have

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\Delta_1}{M}\right)^2 + \left(\frac{\Delta_2}{M}\right)^2 + \ldots + \left(\frac{\Delta_n}{M}\right)^2,$$

in which substitution of the above values gives [69].

By [38] for equal effects we must have

$$\frac{\Delta_1}{M} = \frac{\Delta_2}{M} = \ldots = \frac{\Delta_n}{M} = \frac{1}{\sqrt{n}}\frac{\Delta}{M},$$

from which by substituting the foregoing values of $\frac{\Delta_1}{M}$, $\frac{\Delta_2}{M}$, etc., we obtain the desired expression [70].

Separation into Groups.

$$M = \phi(m_1, \ldots, m_p) \pm \rho(m_q, \ldots, m_s) \pm \ldots \pm \sigma(m_t, \ldots, m_n), \quad [71]$$

where there are p components in the function $\phi(\,)$, r in the function $\rho(\,)$, s in the function $\sigma(\,)$, and n in all.

Resultant Effect,—

$$\Delta^2 = [\Delta\phi(\,)]^2 + [\Delta\rho(\,)]^2 + \ldots + [\Delta\sigma(\,)]^2. \quad [72]$$

Equal Effects,—

$$\frac{1}{\sqrt{p}} \cdot \Delta\phi(\,) = \frac{1}{\sqrt{r}} \cdot \Delta\rho(\,) = \ldots = \frac{1}{\sqrt{s}} \cdot \Delta\sigma(\,) = \frac{\Delta}{\sqrt{n}}, \quad [73]$$

or

$$\Delta\phi(\,) = \sqrt{p} \cdot \frac{\Delta}{\sqrt{n}}, \ \Delta\rho(\,) = \sqrt{r} \cdot \frac{\Delta}{\sqrt{n}}, \ \ldots, \ \Delta\sigma(\,) = \sqrt{s} \cdot \frac{\Delta}{\sqrt{n}}. \quad [74]$$

These serve to determine the values of $\Delta\phi(\,)$, $\Delta\rho(\,)$, etc., from the stated value of Δ. And from them we should proceed to find the corresponding values of δ_1, δ_2, etc., by the general or special formulæ for separate effects.

Deduction.—Adhering to the same notation as elsewhere

$$\Delta^2 = \underbrace{\Delta_1^2 + \ldots + \Delta_p^2} + \underbrace{\Delta_q^2 + \ldots + \Delta_s^2} + \ldots + \underbrace{\Delta_t^2 + \ldots + \Delta_n^2}.$$

In the second member of this expression the first group obviously gives $[\Delta\phi(\,)]^2$, the second $[\Delta\rho(\,)]^2$, the last $[\Delta\sigma(\,)]^2$, so that we may substitute these expressions, obtaining formula [72.]

For equal effects of $\delta_1, \delta_2, \ldots, \delta_n$ in the components, we must have

$$\Delta_1 = \ldots = \Delta_p = \Delta_q = \ldots = \Delta_s = \Delta_t = \ldots = \Delta_n = \frac{\Delta}{\sqrt{n}}.$$

Hence

$$[\Delta\phi(\,)]^2 = \Delta_1^2 + \ldots + \Delta_p^2 = p\left(\frac{\Delta}{\sqrt{n}}\right)^2,$$

and similarly for

$$[\Delta\rho(\,)]^2 = r\left(\frac{\Delta}{\sqrt{n}}\right)^2, \cdot \ldots, \cdot\, [\Delta\sigma(\,)]^2 = s\left(\frac{\Delta}{\sqrt{n}}\right)^2.$$

Hence for equal effects we must have

$$\Delta\phi(\,) = \sqrt{p}\cdot\frac{\Delta}{\sqrt{n}}, \quad \Delta\rho(\,) = \sqrt{r}\cdot\frac{\Delta}{\sqrt{n}}, \quad \ldots, \quad \Delta\sigma(\,) = \sqrt{s}\cdot\frac{\Delta}{\sqrt{n}},$$

as above given; or transposing each term and equating,

$$\frac{\Delta}{\sqrt{n}} = \frac{1}{\sqrt{p}}\,.\,\Delta\phi(\,) = \frac{1}{\sqrt{r}}\,.\,\Delta\rho(\,) = \ldots = \frac{1}{\sqrt{s}}\,.\,\Delta\sigma(\,).$$

$$M = \phi(m_1, \ldots, m_p) \underset{+}{\times} \rho(m_q, \ldots, m_s) \underset{+}{\times} \ldots \underset{+}{\times} \sigma(m_t, \ldots, m_n), \quad [75]$$

where there are p components in $\phi(\,)$, r in $\rho(\,), \ldots,$ s in $\sigma(\,)$.

Resultant Effects,—

$$\left(\frac{\Delta}{M}\right)^2 = \left[\frac{\Delta\phi(\,)}{\phi(\,)}\right]^2 + \left[\frac{\Delta\rho(\,)}{\rho(\,)}\right]^2 + \ldots + \left[\frac{\Delta\sigma(\,)}{\sigma(\,)}\right]^2. \quad [76]$$

Equal Effects,—

$$\frac{1}{\sqrt{p}}\frac{\Delta\phi(\,)}{\phi(\,)} = \frac{1}{\sqrt{r}}\frac{\Delta\rho(\,)}{\rho(\,)} = \ldots = \frac{1}{\sqrt{s}}\frac{\Delta\sigma(\,)}{\sigma(\,)} = \frac{1}{\sqrt{n}}\frac{\Delta}{M}. \quad [77]$$

or

$$\frac{\Delta\phi(\,)}{\phi(\,)} = \sqrt{\frac{p}{n}}\cdot\frac{\Delta}{M}, \quad \frac{\Delta\rho(\,)}{\rho(\,)} = \sqrt{\frac{r}{n}}\,\frac{\Delta}{M}, \ldots, \frac{\Delta\sigma(\,)}{\sigma(\,)} = \sqrt{\frac{s}{n}}\cdot\frac{\Delta}{M}. \quad [78]$$

These determine the values of $\frac{\Delta\phi(\,)}{\phi(\,)}$, etc., whence the cor-

responding values of $\frac{\delta}{m}$ or δ, as the case may require, are computed by the separate effects formulæ.

Example XXII, page 94.

Deduction.—By the general formula [36]

$$\left(\frac{\Delta}{M}\right)^2 = \underbrace{\left(\frac{\Delta_1}{M}\right)^2 + \ldots + \left(\frac{\Delta_p}{M}\right)^2} + \underbrace{\left(\frac{\Delta_q}{M}\right)^2 + \ldots + \left(\frac{\Delta_s}{M}\right)^2} + \ldots$$
$$\ldots + \underbrace{\left(\frac{\Delta_t}{M}\right)^2 + \ldots + \left(\frac{\Delta_n}{M}\right)^2}.$$

To find the values of $\frac{\Delta_1}{M}$, etc., we have

$$\Delta_1 = \frac{df(\,)}{dm_1}\cdot\delta_1 = [\rho(\,) \underset{\div}{\times} \ldots \underset{\div}{\times} \sigma(\,)]\frac{d\phi(\,)}{dm_1}\cdot\delta_1 = \frac{M}{\phi(\,)}\cdot\frac{d\phi(\,)}{dm_1}\cdot\delta_1, \text{ etc.}$$

Dividing both sides by M and passing from the limit to finite changes, we have approximately

$$\frac{\Delta_1}{M} = \frac{\delta_1\phi(\,)}{\phi(\,)}\cdot\frac{\delta_1}{\delta_1} = \frac{\delta\phi(\,)}{\phi(\,)},$$

where $\delta_1\phi(\,)$ denotes the change in $\phi(\,)$ corresponding to δ_1 in m_1. Similarly for the other terms we have

$$\frac{\Delta_p}{M} = \frac{\delta_p\phi(\,)}{\phi(\,)}, \quad \ldots, \quad \frac{\Delta_n}{M} = \frac{\delta_n\sigma(\,)}{\sigma(\,)}.$$

Substituting these values in the above expression we have for the first group

$$\left(\frac{\Delta_1}{M}\right)^2 + \ldots + \left(\frac{\Delta_p}{M}\right)^2 = \left[\frac{\delta_1\phi(\,)}{\phi(\,)}\right]^2 + \ldots + \left[\frac{\delta_p\phi(\,)}{\phi(\,)}\right]^2.$$

But this second member obviously expresses $\left[\frac{\Delta\phi()}{\phi()}\right]^2$. And similarly the second and last groups yield

$$\left[\frac{\Delta\rho()}{\rho}\right]^2 \text{ and } \left[\frac{\Delta\sigma()}{\sigma}\right]^2.$$

Hence for combined effects

$$\left(\frac{\Delta}{M}\right)^2 = \left[\frac{\Delta\phi()}{\phi()}\right]^2 + \left[\frac{\Delta\rho()}{\rho()}\right]^2 + \ldots + \left[\frac{\Delta\sigma()}{\sigma()}\right]^2,$$

which was to be proved.

For equal effects of $\delta_1, \delta_2, \ldots, \delta_n$, we must have

$$\frac{\Delta_1}{M} = \ldots = \frac{\Delta_p}{M} = \frac{\Delta_q}{M} = \ldots = \frac{\Delta_s}{M} = \frac{\Delta_t}{M} = \ldots = \frac{\Delta_n}{M} = \frac{1}{\sqrt{n}}\frac{\Delta}{M},$$

$$\therefore \left[\frac{\Delta\phi()}{\phi()}\right]^2 = \left(\frac{\Delta_1}{M}\right)^2 + \ldots + \left(\frac{\Delta_p}{M}\right)^2 = \frac{p}{n}\left(\frac{\Delta}{M}\right)^2;$$

$$\left[\frac{\Delta\rho()}{\rho()}\right]^2 = \left(\frac{\Delta_q}{M}\right)^2 + \ldots + \left(\frac{\Delta_r}{M}\right)^2 = \frac{r}{n}\left(\frac{\Delta}{M}\right)^2;$$

$$\left[\frac{\Delta\sigma()}{\sigma()}\right]^2 = \left(\frac{\Delta_t}{M}\right)^2 + \ldots + \left(\frac{\Delta_n}{M}\right)^2 = \frac{s}{n}\left(\frac{\Delta}{M}\right)^2.$$

Hence for equal effects we must have

$$\frac{\Delta\phi()}{\phi()} = \sqrt{\frac{p}{n}}\cdot\frac{\Delta}{M}, \quad \frac{\Delta\rho()}{\rho()} = \sqrt{\frac{r}{n}}\cdot\frac{\Delta}{M}, \quad \ldots, \quad \frac{\Delta\sigma()}{\sigma()} = \sqrt{\frac{s}{n}}\cdot\frac{\Delta}{M};$$

or transposing and equating,

$$\frac{1}{\sqrt{p}}\frac{\Delta\phi()}{\phi()} = \frac{1}{\sqrt{r}}\frac{\Delta\rho()}{\rho()} = \ldots = \frac{1}{\sqrt{s}}\frac{\Delta\sigma()}{\sigma()} = \frac{1}{\sqrt{n}}\frac{\Delta}{M},$$

which was to be proved.

Criteria for Negligibility of δ in Components.—Any small change δ_k in any single component m_k is negligible when the corresponding change Δ_k in the result M is

$$\Delta_k \leqq \tfrac{1}{3}\Delta. \quad \ldots\ldots\ldots \quad [79]$$

Any number p of such changes δ_1, δ_2, etc., are negligible when the change corresponding to any one of them is

$$\leqq \frac{1}{3}\frac{\Delta}{\sqrt{p}}, \quad \ldots\ldots\ldots \quad [80]$$

or, more strictly, when

$$\sqrt{\Delta_a^2 + \Delta_b^2 + \ldots + \Delta_f^2} \leqq \tfrac{1}{3}\Delta, \quad \ldots\ldots \quad [81]$$

where Δ_a, Δ_b, etc., are the changes in M corresponding to the changes δ_a, δ_b, etc., in the components m_a, m_b, etc.

The changes Δ_1, Δ_2, Δ_k, etc., in M are, of course, those which enter into the general expression

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_k^2 + \ldots + \Delta_n^2.$$

Thus, in general, $\Delta_1 = \frac{df()}{dm_1}.\delta_1$, etc., or these may have in special cases the special values given in formulæ [33] to [67]; so that we may find Δ_1, etc., having δ_1, etc., given, or may find the value of δ_1 corresponding to the negligible amounts Δ_1, etc.

As we can use precision measures for δ's, the foregoing statements form the criterion by which we may determine when the precision measure of one or of any group of components is negligible in its effect on the *P.M.* of the result.

We have, of course, as a special case of the above general one, δ_k or $\frac{\delta_k}{m_k}$ negligible when

$$\frac{\Delta_k}{M} \leqq \frac{1}{3}\frac{\Delta}{M}, \quad \ldots\ldots\ldots \quad [82]$$

and for p such changes when the effect corresponding to any one of them is

$$\leq \frac{1}{3}\frac{1}{\sqrt{p}}\frac{\Delta}{M}, \quad \ldots\ldots \quad [83]$$

or, more strictly, when

$$\sqrt{\left(\frac{\Delta_a}{M}\right)^2+\ldots+\left(\frac{\Delta_p}{M}\right)^2} \leq \frac{1}{3}\frac{\Delta}{M}. \quad \ldots \quad [84]$$

Example XXIII, page 96.

Deduction.—For the reasons stated at page 17, any single change in Δ which affects it by $\frac{1}{10}\Delta$ or less is negligible. In the expression, then,

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_k^2 + \ldots + \Delta_n^2,$$

what would be the value of any single change, Δ_k, which would reduce Δ to $\Delta - \frac{1}{10}\Delta$, that is to 0.90Δ? The corresponding change in Δ^2 would be

$$\Delta^2 - (0.90\Delta)^2 = \Delta^2 - 0.81\Delta^2 = 0.19\Delta^2;$$

$$\therefore \Delta_k^2 = 0.19\Delta^2, \qquad \Delta_k = 0.44\Delta;$$

$$\therefore \Delta_k = \tfrac{1}{2}\Delta, \text{ approximately.}$$

The limit $\Delta_k = \frac{1}{3}\Delta$ is a safer one and preferable for ordinary work where n is small. It corresponds to a change of $\frac{1}{20}\Delta$. This limit of $\frac{1}{3}\Delta$ is employed in the present demonstration.

If instead of a single change Δ_k there are p changes whose joint effect on Δ is to be considered, then this joint effect would be equal to a single change of the magnitude $\sqrt{\Delta_a^2 + \Delta_b^2 + \ldots + \Delta_p^2}$. For the effect of these on Δ^2 is equivalent to that of a single change Δ_k such that

$$\Delta_k^2 = \Delta_a^2 + \Delta_b^2 + \ldots + \Delta_p^2.$$

This furnishes the general limit, the last one given in the criterion. A special case of this is, of course, when $\Delta_a = \Delta_b = \ldots = \Delta_p$, so that

$$\sqrt{\Delta_a^2 + \Delta_b^2 + \ldots + \Delta_p^2} = \sqrt{p}\Delta_a = \sqrt{p}\Delta_b = \text{etc.}$$

Whence for $\Delta_k = \frac{1}{3}\Delta$ we have

$$\Delta_a = \Delta_b = \ldots = \Delta_p = \frac{1}{3}\frac{\Delta}{\sqrt{p}}.$$

This is a more convenient limit to apply practically, but obviously it is not essential that this equality should exist, so that this is an arbitrary or more special criterion. For reasons which will be shown in the section on equal effects, however, this arbitrary limit will not be widely departed from even under the general limit.

Corresponding criteria can be given for negligible components m. Thus cases will be found where the effect of one of the components m_k itself upon the result M is so small that the component may be neglected altogether in the computation of M and its measurement may be omitted. This would evidently be the case when the omission of m_k did not affect M by an amount exceeding $\frac{1}{10}\Delta$, or better $\frac{1}{20}\Delta$. We should then proceed as follows.

Find the value of δ_k which would be negligible under the criterion $\Delta_k \leqq \frac{1}{3}\Delta$. Then if $m_k \leqq \delta_k$ as thus found, it may be wholly neglected.

If it is a question whether several components may be neglected, then if p is the number of these, the limit for δ_k must obviously be that corresponding to $\frac{1}{3}\frac{\Delta}{\sqrt{p}}$ instead of $\frac{1}{3}\Delta$.

The opportunity for such omissions occurs most frequently in functions containing correction terms, such as temperature corrections or those in the formula for the tangent galvanometer.

Numerical Constants.—A large number of functions contain numerical constants, either mathematical such as π, or physical such as the density of a given substance, etc., whose numerical values are known beyond the requirement of the case. For these we desire to know how many places of significant figures it is necessary to retain in the computation.

The error which enters by them is only that due to the rejected figures, e.g., from using 3.142 for π instead of 3.14 159 265 . . . As the number of such constants in any given function rarely exceeds two, we may safely proceed as follows. From the prescribed or computed value of ΔM calculate what would be a negligible value δc or $\delta c/c$ of the constant just as though it were a directly measured component. Reject all places which do not affect the constant beyond this limit, departing if at all on the side of safety.

Example XXIII, page 96.

Equal Effects.—*Demonstration.* In deducing formulæ [27] etc., the condition was arbitrarily imposed that we should have $\Delta_1 = \Delta_2 = \ldots = \Delta_k = \ldots = \Delta_n = \frac{\Delta}{\sqrt{n}}$, in which case the δ of each component has an equal effect on ΔM. In what way, and to what extent, this is the best guiding condition when δ represents the precision measure may be seen as follows.

If the number, n, of components were indefinitely large, and if the δ's followed strictly the law of deviations, the best values of δ_1, δ_2, etc., for a stated value of Δ, would be such as would render the total labor of observing a minimum. The distribution of precision would thus be determined by the relative difficulty of the several component measurements, and would obviously not be such as to make $\Delta_1 = \Delta_2 =$ etc., that is, it would not correspond to equal effects. The values of the Δ's of the more difficult measurements would be allowed to be greater, and those of the less difficult ones would be made smaller than would be prescribed by that condition. Thus the more difficult measurements would be carried out with a lesser

precision and the easier ones with a greater precision than would correspond to equal effects.

It is, however, not practicable to frame reliable formulæ which shall take into account and give due weight to the difficulty of the component measurements. If such formulæ were attempted, they would be based upon the assumption that n should be large and the law of deviations strictly followed,—conditions which would not be fulfilled by the usual cases in practice, where n is small and the law not closely adhered to. Also it would be necessary to assume some general law as to the increase of labor with increase of precision, and no law would apply with equal exactness to all kinds of measurements.

Thus although in any specific case the condition of equal effects does not yield an exact and complete solution of the best values of the Δ's, yet it will give the best solution to use for the point of departure from which to prescribe the precision of the components. The δ's thus determined should therefore not be regarded at all as absolute and unalterable, but merely as guiding or approximate values from which we should depart slightly in the direction of less precision for the more difficult and greater precision for the less difficult components. Yet it is also clear that we cannot depart widely from this condition. For if we increase the δ of one or more components we must diminish that of some or all of the others in order to preserve the stated value of ΔM, and the limit of this increase is soon reached. Thus to take an extreme case, if Δ_k be the effect on M for the δ of any one component m_k, and if only Δ_k be increased, the largest value which this can have will be $\Delta_k = \Delta M$, which will be only $\sqrt{n}$ times its equal effect value $\Delta M/\sqrt{n}$. And in this case all the other Δ's must be made so small that their joint effect is negligible. Thus if $n = 9$, which is greater than its average value, it would be possible in the extreme case to increase Δ_k to only 3 times its equal effect value.

This extreme limit would rarely be resorted to, first because the conditions which might justify it seldom arise, second be-

cause of an objection to increasing any one or two of the Δ's beyond the others, a reason for which will be given below.

The conditions as to relative difficulty of components which might justify the extreme case would arise only when the labor saved on the one component was equal to the added amount on the others; in other words, when as much labor was required to determine m_k with a *p.m.* producing $\Delta_k = \Delta$ as was necessary to reduce the total resultant effect of the $(n - 1)$ remaining Δ's to a negligible amount, viz. $\frac{1}{3}\Delta$ (formula [80]). Let Δ_e denote the effect of the *p.m.* of each component for equal effects, i.e. $\Delta/\sqrt{n}$. Then this statement means that if all the $(n - 1)$ components are equally difficult, each must have its *p.m.* correspond to $\frac{1}{3}\Delta/\sqrt{n-1}$ or, nearly enough, to $\frac{1}{3}\Delta\sqrt{n}$, i.e. to $\frac{1}{3}\Delta_e$. At the same time Δ_k becomes $\sqrt{n}\Delta_e$. If not all of the $(n - 1)$ components are equally difficult, the latter part of this statement must be modified accordingly.

Ordinarily there would be unequal difficulty among several of the components, so that more than one Δ must be increased and not all the remaining can be made negligible with equal labor. Hence the limit of $\sqrt{n}\Delta_e$ would be rarely called for. The greater the inequality the narrower the allowable limits of change of the Δ's from Δ_e. A change of Δ to $2\Delta_e$ would be unusual.

There is an objection to the increase of any one or two of the Δ's to the extreme limit allowable. For when any Δ is made large compared with most or all of the others the distribution of these quantities as to magnitude ceases to conform at all approximately to the law of deviations and the applicability of all our formulæ based upon that law is thereby materially reduced. The special case when all the Δ's are equal, the δ's being precision measures, conforms closely to that law, owing to the nature of the precision measure.

Estimated Precision Measures of Components. — It is sometimes impracticable to make more than a single measurement of a component at the time of the final measurements, so that there are not data enough for finding its deviation

measure. In such case, it is desirable to take beforehand or afterward a series of observations under similar conditions to obtain those data. For this purpose it is seldom essential that the value of m should be rigidly the same as in the final observation, since the *d.m.* usually changes but little with small changes in m.

It is often impracticable to get any direct data whatever for finding the *d.m.*, but frequently when this occurs an estimate of the *d.m.* can be made in some way which will serve fairly well. One method is to estimate not the *a.d.* but the maximum deviation likely to occur in a large number of deviations, and to divide this maximum by 4, which will give roughly the *a.d.* This is based on the fact that the average frequency of deviations as large as $4a.d.$ is only 1 in 1000, according to the general law of deviations.

Components with Special Laws of Deviation.—Components occasionally are met with whose deviations follow the special law stated at page 21, or other special laws, instead of the general one. The formulæ for the relation of the *P.M.* of the result to the *p.m.* of the components, being based on the general law, are not strictly applicable to indirect measurements containing such components. Nevertheless the inaccuracy introduced by their application to such cases will be hardly, if at all, greater than is very often introduced by the failure of the actual deviations in other components to follow the general law owing to the smallness of the number of the observations, or by the further inaccuracy due to the usually small value of n. It is, therefore, not best to make any exception of these special cases.

Preparation of Functions for Discussion.—This is a point of the utmost importance. Before beginning the precision discussion of any indirect measurement the function $f(m_1, \ldots, m_n)$ must be written out in full. It must be put into such form that *every measured component appears* in it, and the form must be the equivalent of that used for computation. If the latter is done by steps, then all of these must be combined into the general formula. If any of the measured components are re-

lated by equations of condition, these equations do not appear in the expression for $f(\ )$. In problems other than those on best magnitudes the only condition as to independence of the components about which we are concerned is that they shall be obtained by independent measurement.

For example, if we were determining g by the simple pendulum we should, in the simplest case, observe the time t_v of v single vibrations, this number being counted, and we should measure the distance h from the suspension axis to the top of the ball and the vertical diameter d of the ball. The first form which the expression to discuss should receive should not then be

$$g = \pi^2 \frac{l}{t^2};$$

but

$$g = \pi^2 \times \frac{1}{\left(\frac{t_v}{v}\right)^2} \times \left(h + \frac{d}{2} + \frac{2}{5} \frac{\left(\frac{d}{2}\right)^2}{h + \frac{d}{2}} \right),$$

since l is not directly measured, but is computed by means of the expression in the parenthesis.

As however v would be a whole number so small that any miscount would be of the nature of a mistake and not of a deviation, we might very properly not regard v as a measured component, and if we chose could substitute t, the time of a single vibration, for $\frac{t_v}{v}$.

The measured components whose δ's would be discussed would then be t (or t_v), h, and d; not t and l.

After the complete expression has thus been written out it would receive as many simplifications as possible in the course of the discussion, to save labor. The nature of these changes will be further explained.

More extended illustration of this matter will be found in some of the later examples.

Simplification of Functions.—Let the fact be recalled to mind that the solutions of all problems on this subject are to be carried out to only two places of figures and need be correct only to 1 part in 10, and that the results are often (e.g., equal effects) to be regarded only as approximate guides. It will then be seen that simplifications of the original functions will be allowable to save labor, and that they may be of a much more extreme character than would be allowable in the actual computation of the results of the measurement. Moreover, different simplifications may be introduced when different components are being discussed. The one criterion for the admissibility of any proposed simplification, when the solution is being made with respect to any component m_k, is that the value of δm_k or the effect of Δ_k, as the case may be, must not be changed by more than 1 part in 10 by the simplification. Inspection, or a preliminary differentiation, will usually show easily whether the proposed change fulfils this condition.

The two chief methods of simplification are the use of approximate expressions in place of the exact form for $f(\,)$, and the omission of certain terms while differentiating. The tables of approximation formulæ given in some text-books on physical manipulation will suggest available approximations. Many other ways of simplification will be suggested by experience and ingenuity in particular cases.

In somewhat complicated functions one or more components, m_k, etc., may occur in several parts of the expression, as in the complete formulæ for the tangent galvanometer, in the expressions for the efficiency of a dynamo by any of the stray power methods, etc.

If such a case is being treated by the general method, the differentiation may sometimes be simplified either by omitting to differentiate with respect to m_k those terms in which it enters in such a way that the differential will be negligible (less than 0.1 of) compared with the differential of other terms into which m_k enters; or by omitting the differentials of such terms if when made they are found to be thus negligible. Examples of this will be given.

If the case is treated by any of the special formulæ, terms in which m_k enters in this secondary way may be neglected when solving with respect to δm_k. But it must be borne in mind that although one term containing m_k is of negligible (one-tenth) magnitude compared with another, it does not necessarily follow that the effect of δm_k in the one will also be negligible compared with the other. Whether this will be so or not depends upon the form of the function for each of the two terms. Inspection will, however, usually show plainly what terms are negligible.

In separating into groups (pages 61, 62) the groups need not always be strictly exclusive with respect to each other. That is, a given component m_k may be common to two or more groups provided that it occurs only in a secondary way in all groups but one. In discussing such groups m_k would be studied only in the group in which it principally occurred. Sometimes, however, a wider departure than this can be made for the sake of simplifying problems, bearing in mind that the solutions are at best only approximate guides. Thus two or more groups may sometimes be allowed to contain a common component and an allowance made for the fact in the solution and in the result.

Example XXIV.—See also Solutions of Illustrative Problems.

Significant Figures.—In all numerical computations it is essential to know how many significant figures to retain in the data, the results, and at each step in the computations. Failure to follow a correct and consistent practice will result on the one hand in a ludicrous and misleading display of figures and a great waste of time, and on the other, to an ignorant sacrifice of the accuracy of the observations, and to illusive results. Simple rules can be deduced which avoid either extreme. Their application becomes easy and almost involuntary after some practice, although it requires close attention when first undertaken. The following deduction of these rules is in part a special application of the foregoing methods and is given as such.

By a significant figure is here meant any of the ten digits except such zeros as are inserted merely to enable the decimal point to be located. Thus in the number 206.704 every figure would be called a significant figure and there would be said to be six significant figures or six places of significant figures. In 206.70 400 there would be nothing to show whether the last two zeros were significant or useless, but adopting the above definition, as will always be done in these notes, they would be significant figures, of which therefore there would be eight. In 206 700. there would be nothing either in usage or form to indicate whether the last two zeros were significant, i.e. whether there were four, or five, or six significant figures. A direct statement by means of the precision measure or otherwise would be necessary in such a case. In 0.00 206 7, the three zeros preceding the 2 would not be significant and there would be four significant figures. In 0.00 020 670 there would be five significant figures under the usage of the following rules.

It is clear that the number of significant figures is in no way determined or influenced by the position of the decimal point. That point is merely an arbitrary mark to show that the place before it is the place of units.

When either phrase "number of significant figures" or "number of places of significant figures" or "number of places of figures" or "place of figures" or "place" is here used, it will be understood to mean as above explained and to have no reference whatever to the number of decimal places.

This definition of the term "significant figure" is not in accordance with that sometimes given which limits the meaning to the nine digits other than zero. But zero when not used merely to locate the decimal is just as significant as any other digit. For any digit is significant only in the sense that it shows the amount of the stated quantity which exists in the place where the figure stands, and a zero just as truly denotes the amount that exists there as any other digit would do.

Rules for Significant Figures.—Let m denote any numerical result of a measurement either direct or indirect, being either a single observation or a mean result. Let δ denote

either its deviation measure or its precision measure, both based upon the average deviation, viz., on the *a.d.* if m is a single observation, on the *A.D.* if m is a mean.

Rule 1.—In casting off places of figures, increase by 1 the last figure retained, when the following figure is 5 or over.

Example V.—Thus 12.547 becomes 12.55 if one place is rejected, and 12.5 if two are cast off.

If the figure in the last place to be retained happens to be zero it should not be omitted on that account, but the zero should be written in just as any other digit would be, to show that the quantity has been measured or the result carried out to that place.

Example VI.—If one place is to be rejected from 43.704 it would be written 43.70, not 43.7. See also similar instances under other rules.

Rule 2.—In the δ, retain two and no more significant figures.

Example VII.— $\delta = 5200.$, $\delta = 0.085$, $\delta = 3.0$.

Rule 3.—In the quantity m retain enough significant figures to include the place in which the second significant figure of the δ occurs.

Example VIII.—Thus

$m =$	124.032	becomes 124.0	$m =$	762 385.	becomes 762 390.
$\delta =$	1.4	1.4	$\delta =$	680.	680.

Quantities, m, which are obtained by a single observation often fall, unavoidably, one place short of this requirement.

Example IX.—

$$m = 1.875$$
$$\delta = 0.0038$$

Rule 4.—When several quantities are to be added or subtracted.

Find the quantity which has the largest δ, (not δ/m,) and retain in it figures as under Rule 3. In each of the other quantities, strike off all significant figures beyond that corresponding to the last place retained in the one having the largest δ.

Example X.—

Data.		Computation.
No.	*p.m.*	
23.85	0.26	23.9
3 896.0	3.0	3 896.0
— 4.23 37	0.00 47	— 4.2
		3 915.7

Rule 5.—When several numbers are to be multiplied or divided into each other:

Find by inspection the one for which δ/m is greatest. Compute its percentage precision $100\,\delta/m$. If this is

1 per cent or worse, use four significant figures,
$\frac{1}{10}$ " " " " , " five " " ,*
$\frac{1}{100}$ " " " " , " six " " ,

in all data, constant factors, products, quotients, and in the result.

When the Δ/M of the result is computed, any places in M retained under the above rule may be rejected if they exceed the requirements of Rule 3.

Example XI.—To be multiplied together: 29.34 with $\delta = 0.58$, 42 231.6 with $\delta = 1.4$, and $\pi = 3.14\,159\,265$. The first number is clearly the least reliable. Its percentage deviation is $0.58 \div 29. = 2.0$ per cent. Hence we should use four significant figures throughout.

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* It is understood, of course, that this means if $100\frac{\delta}{m} = 1$ per cent and < 10 per cent, and so on for the others.

Ordinary Multiplication.		*Shortened Multiplication.*	
29.34	1 239 000	29.34	1 239 000.
42 230	3.142	42 230.	3.142
8802	2478	11 736	3717
5868	4956	586	1239
5868	1239	58	496
11736	3717	9	24
1 239 028.2	3 892 938.	1 239 000.	3 893 000.
	3 893 000.		

Logarithms.

29.34	1.4675
42 230.	4.6256
3.142	0.4972
3 893 000.	6.5903

In this shortened multiplication the partial products cannot safely be omitted, until one place beyond that to be retained is reached. The method is sufficiently obvious upon inspection. It saves about one third of the time required for such work.

Rule 6.—When logarithms are used, retain as many places in the mantissæ as there are significant figures retained in the data under Rule 5. The characteristic is not to be considered, as it serves merely to locate the decimal point.

Example XII.—See under Rule 5.

Demonstration of Rules.—Rules 1, 2 and 3. The reasons for these have already been given at page 18. Briefly stated they are these: Let the place of m corresponding to that of the second significant figure in its δ be called the rth place. As m is uncertain by an average amount of $\pm\,\delta$, any change which is $\leqq \frac{1}{10}\delta$ may be neglected. The smallest value of $\frac{1}{10}\delta$ will be 1 in the rth which will occur when $\delta = 10$ in the

rth place. Then we may reject all figures in m which do not affect it beyond 1 in the rth place. From the rejection of all places beyond the rth the error in m will not exceed 0.5 in the rth place if Rule 1 be followed. Hence all places beyond the rth should be rejected as called for by Rule 3. The δ is retained to two significant figures chiefly because in computing a ΔM from δ_1, δ_2, etc., or *vice versa*, it is essential to keep the first place surely correct.

Rule 4. Let m' denote that one of the given quantities to be added or subtracted which has the largest precision measure or deviation measure, δ', based on the average deviation; e.g., the number 3 896.0 in Example X. Let the place of figures (not of decimals) in m' corresponding to the second significant figure in δ' be called the rth. Then it is clear, as in the example, that most if not all of the figures in the other quantities which are in places beyond the rth of m' are useless. The question is then just what figures may be stricken out.

Let n be the number of data, some of which may be constants, then the greatest number of rejections to be made will be n. Suppose then that we follow Rule 4, making n rejections by striking out all figures beyond the rth place of m'; what will be the accumulated rejection error in the result?

The law of distribution of the rejection errors is the special law of deviations given at page 21. For we never reject beyond the limits of $+5$ and -5 in the $r+1$st place, and any value between these limits is equally likely to occur at any rejection. The average error for a single rejection is then one half of this limit, viz., 2.5 in the $r+1$st or 0.25 in the rth place. Contrary to the usual case the sign of the rejection error is known for every rejection, and its magnitude also to at least one place of figures. So that if we chose we could compute exactly the actual accumulated rejection error in any given problem. But as we do not care to do this, but wish to deduce a rule which shall be safe to use in all cases, we must assume the accumulation to be according to the formula [24], viz.,

$$E^2 = e_1^2 + e_2^2 + \ldots + e_n^2,$$

where E is accumulated error of result, and e_1, e_2 the errors at each rejection. Then if e is the average rejection error, viz., 0.25 in rth place, we may write

$$E^2 = ne^2 \text{ approx.}$$

This assumes that $e^2 = (e_1^2 + \ldots + e_n^2)/n$, which is a sufficiently close approximation under these conditions. Hence

$$E = e\sqrt{n} = 0.25\sqrt{n} \text{ in } r\text{th place.}$$

Thus if $n = 16$, $E = 1$ unit in rth place, which is negligible in the worst case as shown in demonstration for Rules 1, 2 and 3. Now n would rarely be as great as 16, so that this rule is sufficiently exact for the worst case, and therefore more than close enough for any case of ordinary practice.

It should be noted, however, that E as thus computed is not a definite quantity. It is merely a most probable value of E. But as it is calculated on the same basis as the precision measure δ or Δ, and is therefore a quantity of the same nature, it is proper to use $\frac{1}{10}\delta$ or $\frac{1}{10}\Delta$ in fixing its limit.

Rule 5. This rule may be justified as follows for the case where $\frac{\delta'}{m'} \gtrless 1$ per cent. A similar proof, of course, applies to all cases.

The $\frac{\Delta}{M}$ of the result will be $\gtrless \frac{\delta'}{m'}$ of the least fractionally precise quantity, by formula [52], so that $\frac{\Delta}{M} \gtrless 1$ per cent. In general whatever the value of $\frac{\delta}{m}$, an amount $\frac{1}{10}\frac{\delta}{m}$ will be negligible as being insignificant compared with $\frac{\delta}{m}$. Therefore $\frac{1}{10}\frac{\delta'}{m'}$ and $\frac{1}{10}\frac{\Delta}{M}$ will be negligible. The smallest value of Δ corresponding to this limit will be ± 1 in the fourth place, viz., when $M = 1000$; and we must not allow the accumu-

lated computation error to exceed this amount. This will be accomplished under the Rule 5.

The worst case under this rule is when the first four figures in each factor of the data, constant, intermediate product or quotient, and the final result, are 1000., and if it can be shown that for this the accumulated rejection error does not exceed the limit, the rule will be justified.

Let $m_1, m_2, \ldots, m_n$ denote the values of the various data, constants, and intermediate products or quotients, and suppose a rejection to be made at each of them, viz., n rejections in all, leaving errors $e_1, e_2, \ldots, e_n$. Then

$$\left(\frac{E}{M}\right)^2 = \left(\frac{e_1}{m_1}\right)^2 + \left(\frac{e_2}{m_2}\right)^2 + \ldots + \left(\frac{e_n}{m_n}\right)^2.$$

But when expressed in units in the 4th place, we have under the above conditions

$$m_1 = m_2 = \ldots = m_n = 1000. = M,$$

$$\therefore E^2 = e_1^2 + e_2^2 + \ldots + e_n^2,$$

also expressed in units in the 4th place. Therefore as under Rule 4

$$E^2 = \sqrt{n}e^2 \text{ approx.},$$

where e = average rejection error = 0.25 in 4th place. Thus for $n = 16$

$$E = \pm 1 \text{ in 4th place.}$$

Thus the rule is justified for the worst case, and is therefore sufficient for all.

Here also it is to be remembered that E as thus computed is not a definite but only a most probable value, and is thus a quantity of the same character as δ or Δ, and the criterion of $\frac{1}{10}\frac{\Delta}{m}$ as a limit of negligibility is properly applied. It is not

asserted that the accumulated rejection error will never exceed $E = 1$ in 4th place. On the contrary in 16 rejections an error E of 8 in the 4th place would occur if all the errors e happened to be of the maximum value and in the same direction,—an event which would be exceedingly rare.

Rules which provided that the maximum error should never exceed $\frac{1}{10}\frac{\Delta}{M}$ would be needlessly wasteful of labor. The limit used above is in accordance with that used elsewhere in this book, and is abundantly close for all work except possibly such as conforms with extraordinary closeness to the premises upon which the method of least squares is built up—far more closely than any ordinary physical or technical work.

Rule 6. Inspection of a table of logarithms will show that in the worst case, i.e., where $m = 9999$, a change of 1 unit in the 4th place of the logarithm will correspond to a change of 2 units in the 4th place of the number, which shows that the accumulated error will be about twice as great in the worst case by logarithms under rule 5 as by numbers under rule 4. But on the average the error is very nearly the same by logarithms as by numbers for the same value of n, with the advantage in favor of logarithms as reducing n by requiring fewer intermediate operations.

Forms of Problems on Accuracy of Result.—These are the same as those given for direct measurements at page 33 *et seq.* The procedures there outlined apply to each of the direct measurements from which the indirect result is made up. It is therefore only necessary to add the following.

In an indirect measurement any consideration of the accuracy of a result involves that of each component direct measurement and also involves the discussion of the possible "error of method" (page 46) of the process.

The problems present themselves in these forms.

First. *To obtain by a proposed method the most accurate result practicable.* For this purpose the method as a whole, and the method, apparatus and conditions of work in measuring each component, must be studied as described at page 33.

Second. *To obtain a measurement of the desired quantity, and have the result accurate within a specified limit.* For this the statements made at page 34 apply verbatim, both to the whole method and to the separate components. It is necessary only to add that a preliminary solution by equal effects should be made to determine approximately the best value of the precision measure of each component. Also at the close of the actual observations, it is essential to make a final calculation of the precision measure of the result and to combine with this an estimate, so far as one is possible, of the "error of method" and of the effect of any constant which may be suspected to exist.

Third. *Given a completed result obtained by a stated method, to estimate its accuracy.* The remarks of the corresponding section at page 35 apply equally to indirect measurements.

Data required to Substantiate Result.—The remarks of page 36 apply equally to indirect measurements.

Planning of Indirect Measurement.—To the statements made under (*a*) to (*a*) on page 37 it is only necessary to add that a preliminary precision discussion based on approximate data should invariably be made before a choice of methods is finally reached, or at least before any experimental work beyond preliminary trials is entered upon. The weak or strong points of the proposed methods are often thus developed, and important modifications suggested or errors avoided. It is also of great importance that the plan for the "reduction," i.e., the algebraic and numerical calculations, be thoroughly developed in advance of the measurements. A slight modification of a proposed method may sometimes transform the reduction from a laborious to a much easier one.

EXAMPLES.

Example XIII (see page 50).—The value of g is to be measured by a simple pendulum whose time, t, of a single vibration is to be about 2 seconds; (*a*) what change (in units) in g would correspond to or be produced by a change of 0.1 cm. in l?

$$g = \pi^2 \frac{l}{t^2}.$$

Here g corresponds to M, l to m_1, t to m_2, and π^2 is a constant. We have to find the value of Δ_1 corresponding to $\delta l = 0.1$ cm.

By [19]

$$\Delta_1 = \frac{df()}{dm_1} \cdot \delta_1.$$

$$\frac{df()}{dm_1} = \frac{df()}{dl} = \frac{\pi^2}{t^2} = \frac{3.1^2}{2.^2} = 2.4, \quad \text{and} \quad \delta_1 = \delta l = 0.1.$$

$$\therefore \Delta_1 = 2.4 \times 0.1 = 0.24 \frac{\text{cm.}}{\text{sec.}^2}.$$

(*b*) What change (in units) in g would correspond to a change of 0.02 sec. in t?

$$\frac{df()}{dt} = -2\pi^2 \frac{l}{t^3} = -2 \times 3.1^2 \times \frac{400}{2.^3} = -960.,$$

the length of a 2-sec. pendulum being about 400. cm.

$$\therefore \Delta_2 = -960 \times 0.02 = -19. \frac{\text{cm.}}{\text{sec.}^2}.$$

The negative sign indicates that a + change in t produces a − change in g.

Example XIV (see page 50).—In a measurement of g by a 2-seconds pendulum, as in Example XIII, (*a*) what change in l would produce a change of $1.0 \frac{\text{cm.}}{\text{sec.}^2}$ in g?

By [21]

$$\delta l = \Delta_1 \Big/ \frac{dg}{dl},$$

$$\frac{dg}{dl} = \frac{\pi^2}{l^2} = 2.4, \quad \Delta_1 = 1.0$$

$$\therefore \ \delta l = 1.0/2.4 = 0.42 \text{ cm.}$$

(*b*) What change in t would produce a change of 1.0 per cent in g?

1.0 per cent of g is $0.01g$, and as $g = 980. \frac{\text{cm.}}{\text{sec.}^2}$ nearly, $\Delta g = 0.01 \times 980. = 9.8 \frac{\text{cm.}}{\text{sec.}^2}$.

By [22]

$$\delta t = \Delta g \Big/ \frac{dg}{dt},$$

$$\frac{dg}{dt} = -2\pi^2 \frac{l}{t^3} = -2 \times 3.1^2 \times \frac{400.}{2^3} = -960.$$

$\therefore \ \delta t = 9.8 / -960. = 0.010$ sec. which is $\frac{0.010}{2.0} = 0.005$ or $\frac{1}{2}$ per cent of t.

Example XV (see page 53).—Suppose that in Example XIII the changes $\delta l = 0.1$ cm. and $\delta t = 0.02$ sec. were of the nature of deviations and occurred simultaneously, what would be the resultant effect on g?

By [24]

$$\Delta^2 = \Delta_1^2 + \Delta_2^2,$$

and by Example XIII $\Delta_1 = \frac{df()}{dl} . \delta l = 0.24 \frac{\text{cm.}}{\text{sec.}^2}$, and

$$\Delta_2 = \frac{df()}{dt} . \delta t = -19. \frac{\text{cm.}}{\text{sec.}^2}.$$

$$\therefore \ \Delta^2 = (0.24)^2 + (-19.)^2 = 0.058 + 360. = 360.$$

$$\therefore \ \Delta = \sqrt{360.} = 19. \frac{\text{cm.}}{\text{sec.}^2}.$$

Example XVI (see page 54).—In the measurement of g by a 2-seconds pendulum, as in the foregoing examples, what changes δl in l and δt in t would have a combined effect on g of $\Delta g = 3.0 \frac{\text{cm.}}{\text{sec.}^2}$, supposing them to be of the nature of deviations?

Solving for equal effects by [29] etc., we have

$$\delta l = \frac{\Delta}{\sqrt{2}} \Big/ \frac{dg}{dl} \quad \text{and} \quad \delta t = \frac{\Delta}{\sqrt{2}} \Big/ \frac{dg}{dt}.$$

$$\frac{dg}{dl} = 2.4, \quad \frac{dg}{dt} = -960., \text{ as before, and } \Delta = 3.0$$

$$\therefore \delta l = \frac{3}{1.4} \times \frac{1}{2.4} = 0.88 \text{ cm.}$$

$$\delta t = \frac{3}{1.4} \times \frac{1}{-960.} = -0.0022 \text{ sec.}$$

Example XVII (see page 57).—The rise of temperature $A = t_2 - t_1$ of the water of a continuous calorimeter is to be measured by the reading of two thermometers.

(*a*) What will be the precision of A as affected by t_1 alone if the *p.m.* δt_1 is 0°.02?

By [40],

$$\Delta_1 = \pm \delta_1.$$

$$\therefore \Delta_1 = 0.°02.$$

(*b*) What will be the *p.m.* of A for a *p.m.* of $\delta t_1 = 0°.02$ and $\delta t_2 = 0°.03$?

By [42],

$$\Delta^2 = 0.02^2 + 0.03^2 = 0.00\,04 + 0.00\,09 = 0.00\,13.$$

$$\Delta = 0°.036.$$

(*c*) What will be the *p.m.* necessary in each component for a *p.m.* of 0°.01 in A?

By [43],

$$\delta t_1 = \delta t_2 = \frac{0.01}{\sqrt{2}} = 0°.0071.$$

Example XVIII (see page 58).—An incandescent lamp burns for a time t under a voltage v and with a current c. The quantities c, v, and t are measured in order to determine the amount of heat H produced in this time.

$$H = kcvt,$$

where k = a constant.

(*a*) What *p.m.* in H would correspond to a precision of 0.1 per cent in c?

$$\frac{\delta c}{c} = 0.1 \text{ per cent} = 0.001.$$

By [50] $$\frac{\Delta_1}{M} = \frac{\delta c}{c} = 0.001.$$

The fractional precision in H would then be $\frac{\Delta H}{H} = 0.001$ or 0.1 per cent, whence $\Delta H = 0.001H$ could be found if desired, if H were known.

(*b*) What percentage precision in v alone would correspond to 0.5 per cent in H?

$$\frac{\delta_2}{m_2} = \frac{\Delta_2}{M} = 0.5 \text{ per cent.}$$

(*c*) What would be the fractional precision in H resulting from a fractional precision of $\frac{\delta c}{c} = 0.001$, $\frac{\delta v}{v} = 0.003$, $\frac{\delta t}{t} = 0.002$ in the components?

$$\left(\frac{\Delta}{M}\right)^2 = 0.001^2 + 0.003^2 + 0.002^2 = 0.00\,001\,4;$$

$$\frac{\Delta}{M} = 0.00\,37.$$

(*d*) What percentage precision in each component would correspond under equal effects to 0.2 per cent in H?

$$\frac{\delta c}{c} = \frac{\delta v}{v} = \frac{\delta t}{t} = \frac{1}{\sqrt{3}} \times 0.002 = 0.0012,$$

or 0.12 per cent in each.

Example XIX (see page 59).—The volume of a sphere is to be computed from its measured diameter D. $V = \frac{1}{6}\pi D^3$.

(*a*) If the precision of D is 1 per cent, what is the precision of the result?

$$\frac{\Delta}{V} = 3\frac{\delta}{D} = 3 \times 0.01 = 0.03, \text{ or } 3 \text{ per cent.}$$

(*b*) What precision would be requisite in D for 1 per cent in V?

$$\frac{\delta}{D} = \frac{1}{3}\frac{\Delta}{V} = \frac{1}{3} \times 0.01 = 0.0033 \text{ or } \frac{1}{3} \text{ per cent.}$$

Example XX (see page 60).—For the sake of comparison with former examples take the case of the measurement of g with a 2-sec. pendulum.

(*a*) What would be the precision of the result if the fractional *p.m.* of l were 0.1 per cent, and of t the same?

$$g = \pi^2 \frac{l}{t^2}.$$

We should separate into the factors $\pi^2 \times l \times t^{-2}$. Hence by (*a*)

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta t}{t}\right)^2$$

$$= 0.00\,000\,1 + 0.00\,000\,4 = 0.00\,000\,5,$$

$$\frac{\Delta}{M} = 0.00\,22, \quad \text{or} \quad 0.22 \text{ per cent.}$$

(*b*) What would be the values of δl and δt necessary under equal effects to give g with a precision of 0.001 per cent?

By [67]

$$\frac{\delta l}{l} = 2\frac{\delta t}{t} = \frac{1}{\sqrt{2}} \times 0.001 = 0.00071.$$

$$\therefore \frac{\delta l}{l} = 0.00071, \quad \text{or} \quad 0.071 \text{ per cent},$$

$$\frac{\delta t}{t} = \frac{1}{2} \times 0.00071 = 0.00036, \text{ or } 0.036 \text{ per cent.}$$

To find δl and δt, we must know l and t. t is stated to be 2 sec., and as g must be about $980 \cdot \frac{cm}{sec^2}$, l must be about 4.0 m.

$$\therefore \delta l = 0.00071 \times 400. = 0.28 \text{ cm.}$$

$$\delta t = 0.00036 \times 2.0 = 0.00071 \text{ sec.}$$

Example XXI (see page 62).—The measurement of a current by a cosine galvanometer affords a good example of a function which can be separated into the product of several functions, each of one component only. For a primary cosine galvanometer the formula may be written

$$C = \frac{10Hr}{2\pi n} \cdot \frac{\tan \phi}{\cos \omega}, \quad \ldots \ldots \quad [85]$$

where H = horiz. comp. of earth's field, r = mean radius of coil, n = whole number of turns in coil, ϕ = angle of deflection of needle under the current C, ω = angle of tip of coils from vertical when ϕ is read. The function would be separated as follows:

$$C = \frac{10}{2\pi} \times H \times r \times \frac{1}{n} \times \tan \phi \times \frac{1}{\cos \omega}.$$

Of the components, H and r enter as simple factors, n as a factor to the -1 power, ϕ in the function $\tan\phi$, and ω in the function $\frac{1}{\cos\omega}$ or $(\cos\omega)^{-1}$, both of these functions being factors. For this value of $f(\,)$, the fractional precision would be, by [69],

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\delta H}{H}\right)^2 + \left(\frac{\delta r}{r}\right)^2 + \left(\frac{\delta n^{-1}}{n^{-1}}\right)^2 + \left(\frac{\delta \tan\phi}{\tan\phi}\right)^2 + \left(\frac{\delta(\cos\omega)^{-1}}{(\cos\omega)^{-1}}\right)^2.$$

Suppose a measurement in which the precision of the components was $\frac{\delta H}{H} = 0.001$, $\frac{\delta r}{r} = 0.0005$, $\frac{\delta n}{n} = \text{negl.}$, $\delta\phi = 0°.025$, $\delta\omega = 0°.025$, what would be the value of $\frac{\Delta}{M}$?

From [61] we know that $\frac{\delta n^{-1}}{n^{-1}} = -\frac{\delta n}{n}$. We require further the values of $\frac{\delta\tan\phi}{\tan\phi}$ and $\frac{\delta\cos\omega}{\cos\omega}$. To find these we must have the values of ϕ and ω. The data would furnish this, or in a preliminary discussion typical or limiting values would be used. Suppose $\phi = 45°$ and $\omega = 60°$. Then by [33]

$$\delta\tan\phi = \frac{d\tan\phi}{\tan\phi}\cdot\delta\phi = \sec^2\phi\cdot\delta\phi.$$

$$\therefore \frac{\delta\tan\phi}{\tan\phi} = \frac{\sec^2\phi}{\tan\phi}\cdot\delta\phi = \frac{2\delta\phi}{\sin 2\phi}.$$

To compute the numerical value of this we must have $\delta\phi$ expressed in terms of π.

$$0°.025 = 0.025 \times \frac{\pi}{180} = 0.025 \times 0.017 = 0.00\,043.$$

Hence

$$\frac{\delta\tan\phi}{\tan\phi} = \frac{2 \times 0.00\,043}{\sin 90°} = 0.00\,086.$$

Also

$$\frac{\delta(\cos\omega)^{-1}}{(\cos\omega)^{-1}} = -\frac{\delta\cos\omega}{\cos\omega},$$

$$\delta\cos\omega = \frac{d}{d\omega}(\cos\omega)\cdot\delta\omega = -\sin\omega\cdot\delta\omega$$

$$-\frac{\delta\cos\omega}{\cos\omega} = \frac{\sin\omega}{\cos\omega}\cdot\delta\omega = \tan\omega\cdot\delta\omega.$$

Substituting gives

$$\frac{\delta(\cos\omega)^{-1}}{(\cos\omega)^{-1}} = 1.7 \times 0.00\,043 = 0.00\,073.$$

Then

$$\left(\frac{\Delta}{C}\right)^2 = 0.00\,1^2 + 0.00\,05^2 + 0^2 + 0.00\,086^2 + 0.00\,073^2$$

$$= 0.00\,000\,25.$$

$$\left(\frac{\Delta}{C}\right) = 0.0016.$$

Equal Effects.—What fractional precision in each component would be necessary, δn being negligible, in order that Δ/M should be 0.2 per cent?

For equal effects we must have

$$\frac{\delta H}{H} = \frac{\delta r}{r} = \frac{\delta n^{-1}}{n^{-1}} = \frac{\delta\tan\phi}{\tan\phi} = \frac{\delta(\cos\omega)^{-1}}{(\cos\omega)^{-1}} = \frac{1}{\sqrt{n}}\frac{\Delta}{M}.$$

As before $\dfrac{\delta\tan\phi}{\tan\phi} = \dfrac{2\delta\phi}{\sin 2\phi}$, and $\dfrac{\delta(\cos\omega)^{-1}}{(\cos\omega)^{-1}} = \tan\omega\cdot\delta\omega.$

The number of components is $n = 4$. $\therefore \frac{1}{\sqrt{n}} \frac{\Delta}{M} = \frac{1}{2} \times 0.002$ $= 0.0010$.

$$\therefore \frac{\delta H}{H} = 0.0010;$$

$$\frac{\delta r}{r} = 0.0010$$

$$\frac{\delta \tan \phi}{\tan \phi} = 0.0010 = \frac{2\delta\phi}{\sin 2\phi}.$$

$$\therefore \delta\phi = 0.00050 \sin 90^\circ = 0.00050;$$

$$\delta\phi^\circ = \frac{0.00050}{0.017} = 0^\circ.030;$$

$$\frac{\delta(\cos \omega)^{-1}}{(\cos \omega)^{-1}} = 0.0010 = \tan \omega \,.\, \delta\omega.$$

$$\therefore \delta\omega = 0.0010/\tan 60^\circ = 0.00060.$$

$$\delta\omega^\circ = \frac{0.00060}{0.017} = 0^\circ.035.$$

Example XXII (see page 65).—The following is taken as a simple illustration of the separation into groups. More complex examples will occur in connection with the tangent galvanometer, etc.

A continuous water calorimeter is to be tested by transforming into heat within it a measured amount of electrical energy and measuring this heat by the calorimeter. For instance some incandescent lamps or a coil of wire carrying a current are placed within the calorimeter, the mean current c through the coil and the mean voltage v at its terminals are measured; also the mass m of water passing through the

calorimeter during the measured time τ, and the temperature and t_1 of the entering and outflowing water. Then

$$m(t_2 - t_1) = kcv\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad [86]$$

where k is a constant, viz., the heat-equivalent of 1 watt, calculable from the mechanical equivalent of heat and of the watt. The test consists in ascertaining how closely the experimental value of k, viz.,

$$k = \frac{m(t_2 - t_1)}{cv\tau},$$

agrees with the computed value.

What precision is requisite in each component for a test to 0.1 per cent?

The problem may be solved by the general formula, but could not be solved exactly by the simple formula for factors, since one factor $(t_2 - t_1)$ contains two components. Applying the formula [77] for separation into groups, we have for equal effects

$$\frac{\delta m}{m} = \frac{1}{\sqrt{2}} \frac{\delta(t_2 - t_1)}{(t_2 - t_1)} = \frac{\delta c^{-1}}{c^{-1}} = \frac{\delta v^{-1}}{v^{-1}} = \frac{\delta \tau^{-1}}{\tau^{-1}} = \frac{1}{\sqrt{6}} \frac{\Delta}{M}.$$

Noting that $\frac{\delta c^{-1}}{c^{-1}} = -\frac{\delta c}{c}$, etc., and neglecting signs as of no consequence, we have

$$\frac{\delta m}{m} = \frac{\delta c}{c} = \frac{\delta v}{v} = \frac{\delta \tau}{\tau} = \frac{1}{2.5} \times 0.001 = 0.00\,040,$$

from which the numerical values of δm, δc, δv, and $\delta \tau$ could be computed if m, c, v, and τ were given.

$$\frac{\delta(t_2 - t_1)}{t_2 - t_1} = \sqrt{2} \times 0.00\,040 = 0.00\,056.$$

To find δt_1 and δt_2 we must have the value of $t_2 - t_1$. Suppose this to be given as 10°, then

$$\delta(t_2 - t_1) = 0°.0056.$$

Now by [43], for equal effects

$$\delta t_2 = \delta t_1 = \frac{1}{\sqrt{2}} \cdot \delta(t_2 - t_1) = 0°.0040.$$

Example XXIII (see page 68).—On a cradle dynamometer or on a friction brake the horse-power is given by

$$\text{H.P.} = \frac{2\pi RNW}{33\,000}, \quad \ldots\ldots \quad [87]$$

where R is the radius at which the load W is applied and N is the number of turns per unit of time. N really involves two measured components, viz., a time and a count; but the count is usually made by mechanical means, in such a way that the error of observation resides wholly in the time. We might therefore well substitute for N its equivalent expression $\frac{1}{T}$, where T is the time of a single rotation, and thus find δT. But we may just as well proceed to find δN and from that find δT.

(*a*) Suppose it required to find whether with no other rejection $\frac{\delta W}{W} = \frac{1}{20}$th per cent would be negligible for $\frac{\Delta}{M} = 0.001$. The negligible limit by the criterion would be, as before,

$$\frac{\delta W}{W} \leqq \frac{1}{3} \frac{\Delta}{M} = 0.00\,033,$$

and $\frac{1}{20}$th per cent $= 0.0005$ would not be negligible.

(*b*) Suppose it required to find what values of $\frac{\delta\pi}{\pi}$ and $\frac{\delta W}{W}$ would be simultaneously negligible for $\frac{\Delta}{M} = 0.001$. By the criterion this would be when either was

$$\leqq \frac{1}{3}\frac{1}{\sqrt{p}}\frac{\Delta}{M} = \frac{1}{3}\frac{1}{\sqrt{2}} \times 0.001 = 0.00024.$$

(*c*) Suppose the problem to be, how far must the constant $\frac{2\pi}{33\,000}$ be carried out in order that the rejection error shall be negligible with respect to $\frac{\Delta \text{H.P.}}{\text{H.P.}} = 0.001$. The rules for significant figures would require it to be carried to 5 places, and as π is the only term in it in which a rejection would be made, π must be carried to 5 places of significant figures, viz., to 3.1416. Let us apply the rule of this section and see whether a similar result will be reached.

As only one rejection is to be made, and as π enters as a direct factor, we shall have $\frac{\delta\pi}{\pi}$ negligible when

$$\frac{\delta\pi}{\pi} \leqq \frac{1}{3}\frac{\Delta}{M} = \frac{1}{3} \times 0.001 = 0.0003,$$

or

$$\delta\pi = 0.0003\pi = 0.00093.$$

Carrying π only to 3.142, the rejection only makes an error of $\delta\pi = 0.0004$, which is within the limit assigned by the criterion. Hence by following the criterion we should use $\pi = 3.142$, by the rules for significant figures $\pi = 3.1416$, so that if the criterion is reliable the rules are more than sufficiently precise in this case.

(*d*) Suppose the question to be, would the use for π of 3.142 be admissible with a precision of $\frac{\delta W}{W} = 0.00030$ and a precision of 0.1 per cent required in the result?

By the criterion this would be the case if

$$\sqrt{\left(\frac{\delta\pi}{\pi}\right)^2 + \left(\frac{\delta W}{W}\right)^2} \leqq \frac{1}{3}\frac{\Delta}{M} = 0.00033;$$

$$\frac{\delta\pi}{\pi} = \frac{0.00041}{3.1} = 0.00013;$$

$$\therefore \sqrt{\text{etc.}} = 0.00033.$$

This would be barely negligible.

Example XXIV (see page 76).—The expression for the specific resistance per metre-gramme at 0° c. of a wire may be written in the form

$$S = \frac{m}{l^2} \cdot r' \cdot \frac{1 + \beta_T T}{1 + \beta_\tau \tau} \cdot \frac{1}{1 + \beta_t' t'} \quad . \quad . \quad . \quad [88]$$

where m = mass, l = length, r' = resistance of the sample measured on a Wheatstone bridge, β' = its temperature coefficient, t = its temperature at the time of measurement, β = temp. coeff. of bridge, T = observed temperature of bridge, τ = temperature at which it is correct.

For the precision discussions this expression may be simplified by using the approximations

$$\frac{1}{1 + \beta_\tau \tau} = 1 - \beta_\tau \tau \text{ approx.} \quad \text{and} \quad \frac{1}{1 + \beta_t t} = 1 - \beta_t' t \text{ approx.}$$

These would leave the expression in the form

$$S = \frac{m}{l^2} \cdot r' \cdot (1 + \beta_T T)(1 - \beta_\tau \tau)(1 - \beta_t' t) \text{ approx.},$$

which may be still further simplified by using another approximation and writing

$$S = \frac{m}{l^2} \cdot r' \cdot [1 + \beta_T T - \beta_\tau \tau - \beta_t' t] \text{ approx.}$$

These approximations are well within the limit, for the terms βT, $\beta\tau$, and $\beta' t$ are all less than 0.1, so that the error of the approximation, which is smaller still, is negligible in proportion to 1 in the parentheses.

This final expression affords a good illustration of the method of separation into groups. Counting the temperature coefficients and τ, as we should do in a preliminary discussion at least, where there was any question as to the possibility of their being known accurately enough, we have in the [ ] six components, and in all nine. Hence for equal effects by [78] we should have

$$\frac{\delta m}{m} = 2\frac{\delta l}{l} = \frac{\delta r'}{r'} = \frac{1}{\sqrt{6}} \frac{\delta[\,]}{1} = \frac{.1}{\sqrt{9}} \frac{\Delta}{S},$$

where we write $\delta[\,]/1$ instead of $\delta[\,]/[\,]$, because $[\,] = 1$ sensibly. The precision of the components in the [ ] can thus be much more conveniently studied than by using the general method which would otherwise be necessary.

The omission of terms in differentiating is illustrated in the examples on the tangent galvanometer, cradle dynamometer, and on the Stray Power Test of a dynamo.

BEST MAGNITUDES OF COMPONENTS.

Nature of Problems.—Another class of problems, quite distinct from those which have been discussed, is capable of solution, more or less complete according to circumstances, by the methods which have been developed. These methods have been applied to the calculation of the precision of the result of an indirect measurement from the precision of its components, and to the determination of the best precision of the various components for a specified precision of the result. Beyond these it often happens in designing apparatus or in planning the work of an indirect measurement that such problems as the following are met. Given or having decided upon the apparatus by which the work must be done, and thus the precision with which the components can be measured, it is found that there is some freedom in the proportioning of the parts of instruments, or in the assignment of magnitude to some of the components; and the problem arises to determine what will be the best magnitudes under the conditions of the work. To study some specific problems, suppose that the resistance of a battery is to be measured by the ordinary two-deflection method, using a tangent galvanometer which reads to 0°.1 by a pointer on a graduated circle; what are the best deflections to use, i.e., what two deflections will give the desired result with greatest precision, other things being equal?

Again, suppose there is to be designed a circular cylinder of brass, whose moment of inertia around a transverse central diameter is to be calculable from its mass and measured dimen-

sions, both diameter and length being measured by the same instrument and with the same *p.m.*; what will be the best length and diameter for the cylinder?

Again, what is the best angle at which to use an ordinary tangent galvanometer as far as errors of reading are concerned?

The following statements are expressed in general terms for indirect measurements. Of course problems concerning the design of apparatus, such as that of the cylinder for moment of inertia, fall directly under these statements. Thus in this example the problem is to so construct the bar that its dimensions shall be the best components in the indirect measurement of its moment of inertia.

The possibility of such control of apparatus, method, or magnitude of components by no means always exists. It is frequently precluded by the nature of the process, or by the number of conditions or restrictions placed upon the work. For instance, if the efficiency of a dynamo running under stated conditions is to be measured by a stated electrical method, the magnitude of all the quantities are predetermined by the construction and capacity of the machine, and by the stated conditions of the test. On the other hand, if a certain quantity of heat is to be produced by a current through a wire and this amount is to be computed from measurements of the current, c, resistance, r (or potential), and time, t, there may be best relative magnitudes of c, r, and t for a given set of measuring instruments. It is however less common to meet problems on best magnitudes than on the previous parts of the subject.

The problem of best magnitudes is to some extent a reversal of that of best precision of components, but not strictly so. The data given are the precision of the components, and the problem is to determine the best magnitudes; and to this extent it is the converse of the other proposition. But we have no longer any considerations as to the amount of labor involved or as to its distribution, for that is determined by the precision conditions, which are now fixed. The solution consists, then, merely in finding the relative and numerical

values of some or all of the components which will make Δ a minimum for the given function and precision conditions. If the components are independent quantities, we may solve for the best magnitudes of all of them, if desired. If, however, some or all of them are conditioned, e.g., if one is a function of one or more of the others, the solution cannot be made for all of those thus conditioned; one, at least, of the conditioned quantities must be free, i.e., must be determinable with such a precision that its numerical magnitude may be anything which may be required by the best values of the others by which it is conditioned. When there are three or more components, whether independent or conditioned, it usually occurs that the precision conditions for one or more are such that these quantities can be omitted from the consideration, that is, that their magnitudes may be anything whatever which may be required by the others, thus simplifying the problem.

For a single component the problem takes this form. Having given the form of the function $f(m)$ and the value of δ or δ/m, it is desired to know the best value of m for *a given value of* M.

That this may be solvable, there must occur in $f(m)$ some constant or other quantity which can be changed so that for any value of m the value of M, that is, of $f(m)$, may be made of the specified amount. For instance, in a tangent galvanometer $C = K \cdot \tan \phi$ represents the current C producing a deflection ϕ, K being the factor of the instrument. Here we may wish to know what is the most advantageous deflection ϕ to be used to measure a given current C, in order that we may construct the galvanometer with a suitable value of K.

To solve, we should write an expression for Δ in terms of m, and proceed to find from this the value of m which would make Δ a minimum. This would yield a correct result provided that this expression for Δ did not contain $f(m)$ as a factor. If by dividing through by $f(m)$, or in any other way, it can be shown that the expression for Δ contains $f(m)$, then this factor must be removed before in determining the minimum. For as we wish to find the best value of m for a given value of

M we must determine the value of m which will make Δ a minimum for that value of M, i.e., we must treat M, and therefore $f(\,)$, as a constant in finding this minimum. Now for all functions for which we can write out directly the expression for Δ/M, it is evident that any expression for Δ must be divisible by M. Hence for all such we should save labor by writing the former at once instead of writing out the expression for Δ and dividing by $f(\,)$. It is also clear that if we remove the factor $f(\,)$ from any expression for Δ by dividing the left-hand member by M and the right-hand by $f(m)$, we shall have left an expression for Δ/M, and this it is sometimes possible to do even when we cannot write this expression directly, as will be shown in the example on the tangent galvanometer. It is important to note that the value of m which we find from these expressions for Δ/M is the one which will render the fractional deviation a minimum. If on the other hand we use the expression for Δ the value of m found is that which renders the deviation measure a minimum. In finding the minimum, all constant factors may be omitted, as they do not affect the result.

If the function with which we have to deal is one containing several components of which we are finding the best value of only one, then the components other than m_1 must, of course, be treated as constants.

Briefly then the ***procedure*** to obtain the value of m which will render Δ or Δ/M a minimum for any value of M is as follows:

Write, if possible, the expression for Δ/M for the given function. Otherwise write the expression for Δ, and remove the factor $f(\,)$ if it occurs by dividing by $f(\,)$. Remove all constant factors. Differentiate the resulting expression with respect to m, equate to zero, and solve for m. The criterion that the second differential coefficient must be negative may usually be neglected, inspection serving to determine whether the value found corresponds to a maximum or a minimum.

There is no best value of m for certain functions as follows. For $M = am$ when δ is given, since in that case Δ is a constant

independent of m. For $M = am^t$ when δ/m is given, for $\frac{\Delta}{M} = p\frac{\delta}{m}$, which is independent of m as $\frac{\delta}{m}$ is constant.

Example XXV, page 110.

For two variable components the problem takes this form. Given the form of the function $f(m_1, m_2, \ldots, m_n)$ and the values, either numerical or relative, of δ or δ/m for any two of the variables, e.g., for m_1 and m_2, to find the best ratio of those variables and their best numerical magnitudes. If $n > 2$, then the components other than the two considered must be treated as constants. The following discussion applies only when m_1 and m_2 are independent of each other. The given values of δ or δ/m constitute what may be called the precision conditions. The problem separates naturally into two parts: first, to find the best ratio m_2/m_1; second, to find the best numerical values of m_1 and m_2, to solve which we must previously determine the best ratio. It may occur that only the best ratio is required. We will consider first the finding of the ratio, beginning with the case where δ and not δ/m is given.

Best Ratio.—The best value of m_2/m_1 will be the one which will make Δ a minimum for a given value of M. The procedure must therefore be, first, to obtain a suitable expression for Δ or Δ^2, and then to find by the calculus the value of m_2/m_1 which will make this a minimum.

As to what is a suitable expression for Δ or Δ^2 we may readily see several things. First, it must of course be a function of δ_1, δ_2, m_1, and m_2. If it is a function of δ_1 and δ_2 alone, e.g., $\Delta^2 = \delta_1^2 + \delta_2^2$, it shows that there is no best value for the ratio, for Δ is determined independently of m_1 and m_2. Second, if it should be found to contain $f()$ as a factor, that factor must be omitted in deducing the minimum. For if we desire to find the ratio which will make Δ a minimum for any given value of M we must treat M, and therefore $f()$, as a constant. Now, if we write the expression for Δ^2 for any $f()$, for which we can also write an expression for $\left(\frac{\Delta}{M}\right)^2$ by any of our formulæ, this expression for Δ^2 must be divisible by M^2.

Hence it will be simpler to write at once the expression for $\left(\frac{\Delta}{M}\right)^2$, where the function is such that we can do so, and proceed to find the ratio which will make that expression a minimum. If the expression for Δ^2 has been written and there is any possibility that it contains $f^2()$ as a factor, the test for it should be made by dividing through by it. Any constant factor of the whole expression may be removed, since it is of no effect upon the determination of the minimum.

If instead of δ_1 and δ_2 we have given $\frac{\delta_1}{m_1}$ and $\frac{\delta_2}{m_2}$, there can be no best values for any function for which we can write $\left(\frac{\Delta}{M}\right)^2$ directly in terms of δ_1/m_1 and δ_2/m_2. For from the expression $\left(\frac{\Delta}{M}\right)^2 = \left(p\frac{\delta_1}{m_1}\right)^2 + \left(q\frac{\delta_2}{m_2}\right)^2$, which is the general one for such functions, it is evident that $\frac{\Delta}{M}$ is determined by the given values of $\frac{\delta_1}{m_1}$ and $\frac{\delta_2}{m_2}$ independently of the ratio or values of m_1 and m_2. But with these data the case $\Delta^2 = \delta_1^2 + \delta_2^2$ is soluble, since from the data we have $\delta_1 = \text{const.} \times m_1$ and $\delta_2 = \text{const.} \times m_2$, which will give us an expression for Δ in terms of m_1 and m_2.

The ***procedure*** then briefly stated would be:—Write the expression for $f(m_1, m_2, \ldots, m_n)$ showing properly all the measured components. From this, write out, if possible, the expression for $\left(\frac{\Delta}{M}\right)^2$. Otherwise write the expression for Δ^2 and remove the factor $f^2()$ if it occurs. Remove all constant factors. Find by the calculus the ratio m_2/m_1 which will make the resulting expression a minimum. The cases for which there is no minimum are where $f() = am_1 + bm_2$, given δ_1 and δ_2; and where $f() = a \cdot m_1^p \cdot m_2^q$, given $\frac{\delta_1}{m_1}$ and $\frac{\delta_2}{m_2}$.

To find the minimum, as m_1 and m_2 are independent, it is necessary only to differentiate successively with respect to m_1

and m_2, equate, and solve for m_2/m_1. For the condition for a minimum is that the first differential coefficients shall be simultaneously equal to zero. The further conditions for discriminating between maxima, minima, and points of inflection need not be considered, as inspection will show more easily whether the result obtained corresponds to a minimum.

If m_1 is a function of m_2 and there are but two components, the solution cannot be made for their best ratio, for their ratio is fixed by the function. If, however, m_1 is a function of m_2 and a third component, then the best value of m_2/m_1 may be found, provided that the third component is unrestricted in magnitude. In the solution for best ratio the function must be an expression containing all the measured quantities, just as for all other precision problems. If m_1 is a function of m_2 and m_3, as above supposed, so that $M = f(m_1, m_2, m_3, \ldots, m_n)$ and $m_1 = F(m_2, m_3)$, or $m_2 = F(m_1, m_3)$, or $m_3 = F(m_1, m_2)$, then this function $F()$ is not to be expressed in finding the best ratios, unless it is made use of in computing M from the components. It must, however, be employed in determining the best numerical magnitudes, if the magnitude conditions involve it. Illustrations of this will be seen in the example on moment of inertia.

Best Magnitudes. Two Components.—To find the best numerical values, it is necessary to deduce first the best ratio by the foregoing methods.

As this ratio is the best one for any given value of M, a numerical value of M must be given, if M is a function of only two components m_1 and m_2, before those of the two components can be computed. Thus having given M, $f(m_1, m_2)$, and m_2/m_1, the numerical values of m_1 and m_2 can be found by direct substitution.

Example XXVI, page 111.

If the function is of more than two *independent components*, but we are dealing with only two of them, m_1 and m_2, then in order to compute the best numerical magnitudes we must have given, in addition to the above, the values of the remaining components.

If the function is of more than two components, but m_1 and m_2 are conditioned by a third component, so that $m_1 = F(m_2, m_3)$, then this function $F()$ must be given. In addition to $F()$ we must have given the value of m_3 or such other data as will enable us to compute its value. It is not necessary in this case that M should be given, since $F()$ alone determines the best ratio, and with m_3 the best magnitudes, of m_1 and m_2.

The given value of M, or of m_3 with the function $m_3 = F(m_1, m_2)$, may be called the magnitude conditions, to distinguish them from the given precision conditions under which the best ratio is determined.

Example XXVII, page 112.

For Several Components. *Best Ratios.*—The procedure for three or more components is the same in character as for two. The best ratios are first to be found. For this the proper expression for $f(m_1, m_2, \ldots, m_n)$ in terms of the measured components is written out. From this the expression for $\left(\frac{\Delta}{M}\right)^2$ is written out, if possible; otherwise that for Δ^2 and the factor $f()$ removed by division if it exists. Constant terms which are factors of the whole expression are also removed. The remaining expression is then differentiated successively with respect to m_1, m_2, and so on for all of the components under consideration. All of these coefficients must be simultaneously equal to zero. Hence to find the ratio for any pair of components, we have only to equate the corresponding coefficients and solve for the ratio. It is convenient to find the ratio of each to one common component, e.g., of m_2/m_1, m_3/m_1, etc. Inspection must be resorted to, in order to discriminate between minima and maxima, but this is usually a very simple matter.

The cases for which there is no minimum are the same as for two variables. If some of the components are conditioned by being functions of others, the procedure is the same as for two components, and one free component must exist in each of these functions.

Examples XXVIII, page 115, *and XXIX, page* 118.

Best Magnitudes.—For this part of the solution the require-

ments and procedure are so closely the same as for two components that it is not necessary to restate them. In the substitutions it is convenient to replace all the components by their equivalents in terms of the common component and solve for this; then to obtain all the others by successive substitution in the values for the best ratios.

Approximate Solution by Equal Effects.—Where the number of components is more than one an approximate solution may be used based on the equations for equal effects. Its advantage is greater simplicity and less labor; its disadvantage is that it is only approximate, sometimes only roughly so. With the same values of δ for the components the value of Δ from magnitudes determined by the approximate solution may be several times as great as by the exact solution, but usually it will not be materially greater. It is convenient for complicated functions. It should be remembered that the same arguments which support the method of equal effects in solving for best precision of components do not hold in the present application of it, for the question of the labor involved does not here enter.

Best Ratio.—The procedure is merely the reversal of the equal effects solution for best precision of components given by formula [28]. Having given the form of the function $f(m_1, m_2, \ldots, m_n)$ and the values of δ_1, δ_2, etc., or of $\frac{\delta_1}{m_1}$, $\frac{\delta_2}{m_2}$, etc., the ratios of m_2/m_1, m_3/m_1, etc., which will fulfil the conditions of equal effects, are calculated; and it is *assumed* that these will make Δ approximately a minimum.

Thus the general equation for Δ being

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2,$$

the general equations for equal effects will be

$$\Delta_1 = \Delta_2 = \ldots = \Delta_n = \frac{\Delta}{\sqrt{n}};$$

or, substituting the general values of Δ_1, etc.,

$$\frac{df(\,)}{dm_1}\cdot\delta_1 = \frac{df(\,)}{dm_2}\cdot\delta_2 = \ldots = \frac{df(\,)}{dm_n} = \frac{\Delta}{\sqrt{n}},$$

in which the last term is not needed to find the ratios merely, but may be useful in some cases in finding the best magnitudes. Instead of this general formula we may employ the special one for equal effects which corresponds to the given function.

To make the solution then we have merely to substitute the values of δ_1, δ_2, etc., in the suitable equal effects equations and solve successively for m_2/m_1, m_3/m_1, m_4/m_1, etc.

If the expression $\Delta^2 = \Delta_1^2 + \ldots + \Delta_n^2$ has a factor $f(\,)$, each term must have that factor. And as in the solution for best ratios each term is equated to another, these common factors cancel. Therefore all cases are soluble directly from the expression for Δ and it is not necessary to remove the factor $f(\,)$. A similar inspection shows that the same solution will be arrived at whether we start from the expression for Δ^2 or for $\left(\frac{\Delta}{M}\right)^2$ in a case where the latter is applicable.

Best Magnitudes.—These are determined from the best ratios just as by the exact method.

Example XXX, page 118.

EXAMPLES.

Example XXV.—*Best Magnitudes. One Component.*—Given a tangent galvanometer read by an index moving over a circle graduated in equal parts. Let ϕ be any reading, and K the galvanometer factor. Then the current is

$$C = K \cdot \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [89]$$

The deviation measure, $\delta\phi$, of a single reading will be the same at all parts of the scale. What would be the best deflection to use? This problem might arise from either of the two following questions. For a given current how should K be proportioned in order that the value of ΔC for the given value of $\delta\phi$ should be a minimum? Or if C were variable at will and K were given, what value of ϕ would give C with the greatest fractional precision as far as $\delta\phi$ affects it?

The two problems have the same solution; for, in the first, when ΔC is a minimum $\Delta C/C$ will also be so, as C is a constant. K may be omitted, although it is a component which must be measured, for no limitation is assigned to it in the statement of the problem, and we are therefore to assume it as determinable with any desired precision or with equal precision whatsoever its value.

$$\frac{df(\,)}{dm_1} = K\frac{d\tan\phi}{d\phi} \cdot = K \cdot \sec^2 \phi.$$

$$\therefore \Delta C = K \sec^2 \phi \cdot \delta\phi.$$

Dividing by $f(\,)$ to test whether it is a factor, we have

$$\frac{\Delta C}{C} = \frac{K \cdot \sec^2 \phi}{K \cdot \tan \phi} \cdot \delta\phi = \frac{1}{\sin \phi \cdot \cos \phi} \cdot \delta\phi = \frac{2}{\sin 2\phi} \cdot \delta\phi,$$

which shows that the factor $f(\,)$ exists and leaves us, as the expression to be made a minimum, omitting the constant factor 2,

$$\frac{1}{\sin 2\phi}.$$

It is easy to see by inspection that this is a minimum for $\phi = 45°$; but proceeding with the general method, we have

$$\frac{d}{d\phi}\cdot\frac{1}{\sin 2\phi} = \frac{d}{d\phi}(\operatorname{cosec} 2\phi) = -2\frac{\cos 2\phi}{\sin^2 2\phi} = 0.$$

$$\therefore \cos 2\phi = 0, \quad \text{and} \quad \phi = 45°,$$

as the best value of ϕ, answering either requirement.

***Example XXVI.**—Best Magnitude. Two Components.*—The rate h of production of heat in a conductor is to be determined by measuring the current c and the voltage v. The instruments available can measure the current with a precision of $\delta c = 0.05$ ampere, and the voltage to $\delta v = 0.05$ volt. What are the best values of c and v for a rate of heating of 25 calories per second. For kgm., deg. C., true volts and ohms we have, at a temperature of 15° C.,

$$h = 0.2387\, vc. \quad . \quad . \quad . \quad . \quad . \quad . \quad [90]$$

The conditions are $\delta v = 0.05$, $\delta c = 0.05$, $h = 25$.

There are only two variable components, $m_1 = v$, $m_2 = c$; and both are to be discussed. As the resistance of the conductor is not specified, they are independent. Expression [90] for $f(\,)$ contains all the measured quantities, and is such that we can write the expression for $\left(\frac{\Delta}{M}\right)^2$, viz.,

$$\left(\frac{\Delta}{M}\right)^2 = \left(\frac{\delta v}{v}\right)^2 + \left(\frac{\delta c}{c}\right)^2 = \frac{\delta^2 v}{v^2} + \frac{\delta^2 c}{c^2}.$$

Then

$$\frac{d}{dv}\left(\frac{\Delta}{M}\right)^2 = -2\frac{\delta^2 v}{v^3}, \qquad \frac{d}{dc}\left(\frac{\Delta}{M}\right)^2 = -2\frac{\delta^2 c}{c^3}.$$

Equating to zero simultaneously and solving we have

$$-2\frac{\delta^2 v}{v^3} = -2\frac{\delta^2 c}{c^3}; \qquad \therefore \frac{c^3}{v^3} = \frac{\delta^2 c}{\delta^2 v} = \left(\frac{0.05}{0.05}\right)^2 = 1.$$

$\therefore \frac{c}{v} = 1.$ is the best ratio.

To find the best numerical magnitudes we may substitute this ratio and the value of h in the original expression and solve. Thus

$$h = 0.24\, v \times v = 0.24\, v^2;$$

$$\therefore v = \sqrt{\frac{25.}{0.24}} = 10. \text{ volts};$$

$$c = v = 10. \text{ amperes.}$$

This problem is purposely made numerically simple so that inspection may readily show the results to be correct.

Example XXVII.—*Best Magnitudes. Two out of Three Components.*—A bar is to be constructed whose moment of inertia may be computed from its measured mass m and its linear dimensions. It is to be a right circular cylinder of height h and diameter d, and is to swing about a transverse central diameter. Both h and d are to be measured with the same micrometer screw, but owing to the uncertainty from rounded edges and other imperfections of the ends $\delta h = 4\delta d$. The mass m is to be found by weighing, and can be determined so closely that $\delta m/m$ is negligible. What is the best ratio of d/h?

Here there are three measured components, but one of them, m, is omitted from the discussion because its $\delta m/m$ is negligible. This, moreover, is a function of the other two components, so that unless it or one of the others could be omitted, the problem could not be solved. The expression for the moment of inertia is

$$I = m\left(\frac{h^2}{12} + \frac{d^2}{16}\right). \quad \ldots\ldots \quad [91]$$

The function $f(\,)$ is such that we cannot write out $\left(\frac{\Delta}{M}\right)^2$ for it by any of the special formulæ. Applying the general formula, we have

$$\Delta^2 = \left(\frac{df(\,)}{dh}\cdot\delta h\right)^2 + \left(\frac{df(\,)}{dd}\cdot\delta d\right)^2;$$

$$\frac{df(\,)}{dh} = \frac{mh}{6}, \quad \frac{df(\,)}{dd} = \frac{md}{8}.$$

$$\therefore \Delta^2 = \frac{m^2h^2}{36}\cdot\delta^2 h + \frac{m^2d^2}{64}\cdot\delta^2 d.$$

This is not divisible by the expression $f(\,)$, hence we cannot take out $f(\,)$ as a factor. Omitting then the common constant factor $m^2/4$, we have as the expression to be made a minimum

$$\frac{h^2}{9}\cdot\delta^2 h + \frac{d^2}{16}\cdot\delta^2 d,$$

which will be denoted by [ ]. Then

$$\frac{d[\,]}{dh} = \frac{2}{9}h\cdot\delta^2 h = 0, \quad \frac{d[\,]}{dd} = \frac{1}{8}d^2\cdot\delta^2 d = 0.$$

$$\therefore \tfrac{2}{9}\,h\cdot\delta^2 h = \tfrac{1}{8}\,d\cdot\delta^2 d.$$

$$\frac{d}{h} = \frac{16}{9}\cdot\frac{\delta^2 h}{\delta^2 d} = \frac{16}{9}\cdot\frac{4^2}{1^2} = \frac{256}{9} = 28.$$

The best ratio then is $d/h = 28$, that is, the cylinder should be a very short one, a disc rather than a long cylinder. This result is rather striking, as the first thought might be that δh being greater than δd, h should be made greater than d, so that the fractional accuracy of h should be increased. Further consideration and inspection of the formula for I will, however, show that the result arrived at is rational under the conditions of $\delta h = 4\,\delta d$. Moreover, if the value of Δ^2 be computed for this ratio, and then for another ratio quite different, but which makes the value of I the same, the value of Δ will be found to be greater in the second case. A test of that sort will also

show another thing which has elsewhere been noted, namely, that the value of Δ will change but little for considerable changes in the ratio. This is, however, dependent on the form of $f(\,)$.

To find the best numerical magnitudes for h and d, we may have the necessary magnitude conditions given in several ways; two examples will be taken.

First. The most usual form of the problem would be this. To find the best values of h and d for a bar of stated moment of inertia and material; for example, moment of inertia to be 1500 c. g. s., and the bar to be of brass. Since $m = \frac{1}{4}\pi d^2 h\rho$, where ρ is the density, we must know ρ in order to find what m would be. Suppose for the brass $\rho = 8.5$. The magnitude conditions would then be $I = 1500$ and $\rho = 8.5$; and there would be the equation $m = \frac{1}{4}\pi d^2 h\rho$ conditioning m, h, and d.

To find the best value of d we may substitute the magnitude conditions and the value $h = d/28$ in the full expression for I, viz.,

$$I = \frac{1}{4}\pi d^2 h\rho\left(\frac{h^2}{12} + \frac{d^2}{16}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [92]$$

$$\therefore 1500 = \frac{1}{4} \times 3.1 \times d^2 \times \frac{d}{28} \times 8.5\left(\frac{d^2}{28^2 \times 12} + \frac{d^2}{16}\right).$$

$$\therefore d^5 = 100\,000. \quad d = 10. \text{ cm.}; \quad h = 10/28 = 0.36 \text{ cm.}$$

Second. It might be required to construct the bar of brass and with a stated mass, e.g., $m = 63.$ gms. The magnitude conditions would then be $m = 63.$ gms., $\rho = 8.5$, and there would be the same condition equation $m = \frac{1}{4}\pi d^2 h\rho$.

Then d and h must fulfil these conditions and have the ratio $d/h = 28$, simultaneously. We must therefore eliminate d and h successively between

$$63 = \frac{1}{4}\pi d^2 h \times 8.5, \quad \text{and} \quad \frac{d}{h} = 28.$$

$$\therefore 63 = \tfrac{1}{4} \times 3.1 \times 28^2 h^2 \times h \times 8.5.$$

$$h^3 = 0.0122, \quad h = 0.35 \text{ cm.}, \quad d = 28h = 9\,8 \text{ cm.}$$

These results of course agree with those of the first case within the limits of error of two-place computations, the data being equivalent.

Example XXVIII.—Best Magnitudes. Several Components. —The modulus of elasticity E of an unplaned wooden beam 10 ft. long is to be determined by measuring the weight W at its centre necessary to produce a transverse central deflection v when supported at the ends, and by measuring the mean breadth b, depth h, and the length l. The value of E is known in advance to be about 1.3×10^6 lbs. per sq. in., and examination of the beam and measuring apparatus shows that the precision attainable will be $\delta l = 0.5$ in., $\delta b = \delta h = 0.05$ in., $\delta v = 0.002$ in., $\frac{\delta W}{W} = 0.001$. Desired the best magnitudes of the components.

The expression for E is

$$E = \frac{1}{4}\frac{Wl^3}{bh^3v} \quad \ldots\ldots\ldots \quad [93]$$

The components are connected by no equation of condition among themselves, so that the best ratios might be found for all of them if the precision conditions permitted, were it not for the fact that the length l is specified. But we can write the expression for $\left(\frac{\Delta}{M}\right)^2$ directly by [57], viz.,

$$\left(\frac{\Delta}{M}\right)^2 = \frac{\delta^2 W}{W^2} + 9\frac{\delta^2 l}{l^2} + \frac{\delta^2 b}{b^2} + 9\frac{\delta^2 h}{h^2} + \frac{\delta^2 v}{v^2}.$$

Now as $\frac{\delta W}{W}$ is a constant by the conditions, its effect on M will be the same whatever the value of W; hence there will be no best magnitude for W, and it may be omitted from the discussion. This fact enables us to introduce the condition

$l = 120$ in. Proceeding then with the others to find the best ratios, we have

$$\frac{d()}{dl} = -18\frac{\delta^2 l}{l^3}, \quad \frac{d()}{db} = -2\frac{\delta^2 b}{b^3},$$

$$\frac{d()}{dh} = -18\frac{\delta^2 h}{h^3}, \quad \frac{d()}{dv} = -2\frac{\delta^2 v}{v^3}.$$

Hence

$$-18\frac{\delta^2 l}{l^3} = -2\frac{\delta^2 b}{b^3}, \quad \frac{b^3}{l^3} = \frac{1}{9}\cdot\frac{\delta^2 b}{\delta^2 l} = 0.0011. \qquad \therefore \frac{b}{l} = 0.10.$$

$$-18\frac{\delta^2 l}{l^3} = -18\frac{\delta^2 h}{h^3}, \quad \frac{h^3}{l^3} = \frac{\delta^2 h}{\delta^2 l} = 0.010. \qquad \therefore \frac{h}{l} = 0.22.$$

$$-18\frac{\delta^2 l}{l^3} = -2\frac{\delta^2 v}{v^3}, \quad \frac{v^3}{l^3} = \frac{1}{9}\cdot\frac{\delta^2 v}{\delta^2 l} = 0.00000018. \quad \therefore \frac{v}{l} = 0.012.$$

To find the *best magnitudes*, $l = 120.$ in.

$$\therefore b = 0.10\, l = 12. \text{ in.};$$

$$h = 0.22\, l = 26. \text{ in.};$$

$$v = 0.012\, l = 1.4 \text{ in.}$$

Note that as the components are all independent, and as W may have any value as far as the conditions show, we should be obliged to assign a value to some component, if not to l.

To see whether these best magnitudes are practicable we should compute W to see whether it came within the limits of the testing-machine. Transposing and substituting gives

$$W = \frac{4bh^3vE}{l^3} = \frac{4 \times 12 \times 26^3 \times 1.4 \times 1.3 \times 10^6}{120^3} = 1.1 \times 10^6.$$

But 1 100 000 lbs. would be beyond the limits of most machines, so that we should be obliged to use some smaller values

for b, h, and v, and should wish to maintain the same ratios. Let us then limit the load to $W = 100\,000$. lbs. We must therefore have

$$bh^2v = \frac{Wl^3}{4E} = \frac{10^5 \times 120^3}{4 \times 1.3 \times 10^6} = 3.3 \times 10^4;$$

also

$$\frac{b}{h} = \frac{0.10}{0.22} = 0.45, \quad \frac{v}{h} = \frac{0.012}{0.22} = 0.055.$$

Substituting the latter gives

$$0.45 \times 0.055h^3 = 3.3 \times 10^4. \quad \therefore 0.025h^3 = 3.3 \times 10^4.$$

$$\therefore h = 17. \text{ in.};$$

$$b = 0.45 \times 17. = 7.7 \text{ in.};$$

$$v = 0.055 \times 17. = 0.94 \text{ in.}$$

This result may be checked by substituting the values in the expression for E, which should yield 1.3×10^6 within one or two units in the second place.

To see how much difference there would be in the precision of E under the two cases we may compute Δ/M for each.

For the first values,

$$\left(\frac{\Delta}{M}\right)^2 = 0.001^2 + 9\left(\frac{0.5}{120.}\right)^2 + \left(\frac{0.05}{12}\right)^2 + 9\left(\frac{0.05}{26}\right)^2 + \left(\frac{0.002}{1.4}\right)^2$$

$$= 0.00\,000\,1 + 0.00\,014 + 0.00\,001\,6 + 0.00\,003\,6 + 0.00\,000\,3$$

$$= 0\,00\,020.$$

$$\frac{\Delta}{M} = 0.014.$$

For the second values,

$$\left(\frac{\Delta}{M}\right)^2 = 0.001 + 9\left(\frac{0.5}{120.}\right)^2 + \left(\frac{0.05}{7.7}\right)^2 + 9\left(\frac{0.05}{17.}\right)^2 + \left(\frac{0.002}{1.4}\right)^2$$

$$= 0.00\,000\,1 + 0.00\,014 + 0.00\,004\,2 + 0.00\,008\,1 + 0.00\,000\,3$$

$$= 0.00\,026.$$

$$\frac{\Delta}{M} = 0.016.$$

The close agreement of these two values shows that the second set of magnitudes is sensibly as good as the first.

***Example XXIX.**—Best Magnitudes. Conditioned Components.*—The specific resistance per metre-gramme of a sample of copper wire is to be determined by measuring the resistance, R, and mass, m, of a measured length l of the wire. Given $\delta R = 0.001$ ohm, $\delta m = 0.001$ grm., $\delta l = 0.3$ mm.

In this form the problem is insoluble. For as the material of the wire is stated, the mass is a function of the length, diameter, and specific gravity, and the resistance is also a function of the length, diameter, and resistance per unit volume, and as both specific gravity and resistance per unit volume are constants, R, m, and l are conditioned quantities. This may be expressed in another way by saying that as the material is fixed, then for any given value of m/l, that is, of mass per unit length, there is a fixed value of R/l, that is, of resistance per unit length. We are therefore not at liberty to assign ratios $R : m : l$ on the basis of the precision conditions. It could be solved for $R : l$ under the condition δm negligible.

The above form of the precision conditions is not, however, the one which would ordinarily arise in practice. We should usually have given $\delta R/R$, and $\delta l/l$, that is, R and l would each be measurable with a constant fractional precision. And $\delta m/m$ would be usually negligible, measurements by the balance available being generally far more precise than the measurements of either R or l. In this case there would also be no best ratios, since Δ/M is a constant, being fixed by the precision conditions independently of the values of the components. The best values would be determined by other limitations of the apparatus.

***Example XXX.**—Best Magnitudes. Equal Effects.*—In illustration of this method the preceding example on the determination of E for a wooden beam will be taken. Applying the method we have

$$\frac{\delta W}{W} = 3\frac{\delta l}{l} = -\frac{\delta b}{b} = -3\frac{\delta h}{h} = \frac{\delta v}{v}.$$

Then for $\frac{\delta W}{W}$ constant, we have for the *best ratios*

$$3\frac{\delta l}{l} = -\frac{\delta b}{b}, \quad \frac{b}{l} = -\frac{1}{3}\cdot\frac{\delta b}{\delta l} = -0.033;$$

$$3\frac{\delta l}{l} = -3\frac{\delta h}{h}, \quad \frac{h}{l} = -\frac{\delta h}{\delta l} = -0.10;$$

$$3\frac{\delta l}{l} = -\frac{\delta v}{v}, \quad \frac{v}{l} = -\frac{1}{3}\cdot\frac{\delta v}{\delta l} = -0.0013.$$

The negative signs indicate merely that an increase in that component causes a decrease, or a negative error, in the result.

For best magnitudes with $l = 120$ in. we then have

$$b = 0.033\ l = 4.0 \text{ in.};$$

$$h = 0.10\ l = 12. \text{ in.};$$

$$v = 0.0013\ l = 0.16 \text{ in.}$$

These dimensions would require a load

$$W = 4\frac{1.3 \times 10^6 \times 4 \times 1700 \times 0.16}{1.7 \times 10^6} = 3300. \text{ lbs.},$$

which is small for the capacity of the machine. One objection to the dimensions would be that they are too small to correspond to commercial sizes. They would be modified as in the former example.

SOLUTIONS OF ILLUSTRATIVE PROBLEMS.

Example XXXI.—*Problem.*—A voltmeter is to be calibrated by the Poggendorff method, using a Carhart-Clark cell. The calibration at 110 volts is desired to 0.2 volt, of which error one half is allowable in the cell measurement, the other 0.1 volt being assigned to the voltmeter. The arrangement used is to connect the voltmeter in series with a battery of over 110 volts, a controlling water rheostat, and an accurate adjustable resistance r, the Clark cell being connected around this resistance. The water rheostat is adjusted until the voltmeter reads 110 volts, and the resistance r until on closing a key in the cell circuit no deflection occurs on a sensitive galvanometer in that circuit, an approximate adjustment being effected at first by a preliminary computation, in order to avoid injury to the cell.

Solution.—Let R denote the voltmeter resistance; r, the adjustable resistance; t, the observed temperature of the cell $E = 1.438$ legal volts (1884), the voltage of the cell at 15° C.; $a = -0.00038$, the temperature coefficient of the cell; then the voltage at the terminals of the voltmeter will be

$$V = E[1 - a(t - 15)]\frac{R}{r}. \quad . \quad . \quad . \quad . \quad . \quad [94]$$

By computation, if $V = 110$, $R = 17000$, and $t = 15°$, we find $r = 220$ ohms approximately, which would be about the amount which we should require to obtain the balance.

To find the accuracy requisite in the 5 components E, a, t, R, and r, we apply the general formula [38]. We will assume $t = 20°$ as a typical value.

$$\frac{dV}{dE} = \frac{R}{r} \qquad = \frac{17\,000.}{220.} = 80;$$

$$\frac{dV}{da} = -E\frac{R}{r}(t - 15) = -1.4 \times 80 \times (20 - 15) = -560.;$$

$$\frac{dV}{dt} = -Ea\frac{R}{r} \qquad = -1.4 \times 80 \times 0.00\,038 = -0.042;$$

$$\frac{dV}{dR} = \frac{E}{r} \qquad = 1.4/220 = \frac{1}{160};$$

$$\frac{dV}{dr} = -E\frac{R}{r^2} \qquad = -1.4 \times \frac{80}{220} = -\frac{1}{2}.$$

$\Delta V = 0.10$ volts. $\Delta V/\sqrt{n} = 0.10/\sqrt{5} = 0.045$ volts.

$\delta E = 0.045/80. = 0.00\,056v$. Attainable. There is probably a constant error of more than half of this amount in the absolute value of the ohm.

$\delta a = 0.045/(-560) = -0.00\,008\,0$. Easily made negligible.

$\delta t = 0.045/(-0.042) = -1°.1$. " " "

$\delta R = 0.045 \times 160 = 7.2$ ohms $= 0.04$ %.
$\delta r = 0.045 \times (-2) = -0.090$ ohms $= 0.04$ %.
} As only the ratio of $R : r$ is required, the necessary precision may be reached; but with German-silver coils a variation of 1° in temperature would correspond to this amount.

Thus the precision measure attainable would be about

$$\left(\frac{\Delta}{V}\right)^2 = \left(\frac{\delta_1}{E}\right)^2 + \left(\frac{\delta_4}{R}\right)^2 + \left(\frac{\delta_5}{r}\right)^2 = 3 \times (0.00\,04)^2.$$

$$\frac{\Delta}{V} = 0.000\,68 = 0.068 \text{ per cent},$$

which is slightly better than the 0.1 per cent called for, which may therefore presumably be attained.

Example XXXII.—Efficiency of Electric Generator or Motor by Stray-power Method.

Explanation of Method.—If in a generator we let L denote the total power losses in the machine, whether electrical or mechanical, and if we let the mechanical power applied to the machine, i.e., the input, be denoted by I, and the electrical power available in the circuit outside of the machine, in other words the electrical output, by O, then the "commercial efficiency" of the generator is

$$E = \frac{O}{I}, \quad \ldots\ldots\ldots \quad [95]$$

and the total power losses are

$$L = I - O, \quad \ldots\ldots\ldots \quad [96]$$

all power being, of course, expressed in the same unit. Whence

$$I = O + L, \quad \ldots\ldots\ldots \quad [97]$$

and

$$E = \frac{O}{O + L}. \quad \ldots\ldots\ldots \quad [98]$$

From this last expression it is obvious that if we have any means of measuring the losses L we may ascertain the efficiency from measurements of L and O alone without measuring I. Several methods exist by which this can be done, some of which measure L by electrical, others by mechanical methods and still others by a combination of the two.

The above formula for E for generators must, of course, be slightly modified to be applicable to motors. Let i be the electrical input or electrical power supplied to the motor, and o its mechanical output. Then its commercial efficiency is

$$e = \frac{o}{i}. \quad \ldots\ldots\ldots \quad [99]$$

Then, as before, if l denotes the total losses in the machine, we have

$$l = i - o; \quad \therefore o = i - l, \quad . \quad . \quad . \quad . \quad [100]$$

and

$$e = \frac{i-l}{i} = 1 - \frac{l}{i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad [101]$$

whence by measuring i and l we may calculate e. As for the generator so for the motor there are several methods by which l may be measured electrically or mechanically.

Of the electrical methods for generators and motors the simplest is the "Stray-power" method. This is applicable to a wide variety of machines, and forms perhaps the best available method for general technical testing. In common with all methods which measure the losses, it possesses an important advantage over those methods which measure both o and i. For inspection of the expression for e will show that an error of 1 per cent in the losses corresponds only to about 0.1 per cent in e since l is only about one tenth of i, and a similar statement is true for a generator. The discussion will also show that the number of components to be measured becomes so small that this fact in combination with the above enables the method to give a considerably greater precision in the efficiency than is required for any of the components. Briefly explained the method is as follows:

The losses of power in a dynamo (either motor or generator) may be divided into, 1st, the loss, A, due to the armature resistance; 2d, that, F, due to the field resistance; 3d, the sum total of the losses due to all other sources, viz., Foucault currents, hysteresis, bearing friction, air resistance, etc. The third group has been termed the "stray power," and will be denoted by SP. For a stated condition of running, that is, at a specified voltage, current, and speed, each of these losses has a fixed value provided that the condition of steady temperature of the machine has been reached.

The first two losses, A and F, can be calculated if the following quantities are known, viz.,

c_a = current in armature,
c_f = " " field coils;
r_a = resistance of armature;
r_f = " " field coils,
or v_f = voltage between field terminals.

For

$$A = c_a^2 r_a,$$

$$F = c_f^2 r_f, \quad \text{or} \quad F = \left(\frac{v_f}{r_f}\right)^2 r_f = \frac{v^2}{r_f},$$

where either c_f or v_f may be used as is most convenient in any given case.

In what follows let us for simplicity suppose that we have a simple shunt-wound motor.

The stray power cannot be directly measured, but must be indirectly determined by difference. We have, of course,

$$L = A + F + SP, \quad . \quad . \quad . \quad . \quad . \quad [102]$$

Thus if L, A, and F are measured under any given condition, SP for that condition can then be deduced.

The stray-power method rests upon this latter fact. It also involves the facts that the SP does not change widely between full load and no load on most types of machine, that it is of sensibly the same amount for the same machine whether acting as a generator or motor, and that its change with slightly different speeds of the machine may be allowed for approximately. The procedure consists in running the motor with no load under as nearly as possible its rated voltage and speed, and measuring the actual voltage v, current c, and speed s. The armature resistance r_a, the field resistance r_f, and the current c_f or the voltage v_f between the field terminals, for the same conditions must also be measured or otherwise ascertained, except as shown below. Then the total loss of power under this condition is

$$l = cv,$$

and the stray power is

$$sp = l - (a + f) = cv - c_a^2 r_a - c_f^2 r_f \quad . \quad . \quad [103]$$

Let SP denote the stray power of the machine under any specified load at rated speed S and voltage V. Then it is assumed, first, that $SP = sp$ sensibly if $S = s$, and, second, that if s differs from S, then

$$SP = \frac{S}{s} . sp. \quad . \quad . \quad . \quad . \quad . \quad . \quad [104]$$

Both assumptions are fairly well supported by the comparison of results of experimental tests of the same machine by different methods, but neither is exact, nor is either reliable for machines of inferior design. The assumptions may probably be regarded as introducing an error of less than one half of one per cent into E under ordinary conditions of testing. Hence the efficiency of the given machine under the specified load may now be calculated. Let V be the rated voltage, C the current corresponding to the specified load, and S the rated speed; then we have

$$\begin{aligned} l &= A \quad + F \quad + SP \\ l &= C_a^2 R_a + C_f^2 R_f + \frac{S}{s}(cv - c_a^2 r_a - c_f^2 r_f); \\ i &= CV; \\ e &= 1 - \frac{l}{i} = 1 - \frac{1}{CV}\left\{ C_a^2 R_a + C_f^2 R_f + \frac{S}{s}(cv - c_a^2 r_a - c_f^2 r_f) \right\}. \quad [105] \end{aligned}$$

Here the capitals denote the quantities for the specified load, and the small letters for the measurement with no load, i.e., when the stray power is being measured. The quantities C, V, and S are not measured, they are merely specified amounts. The quantities R_a and R_f must be determined by measurement or otherwise for the specified condition, and likewise C_f, except as shown below. There are therefore at most nine quantities, viz., R_a, R_f, s, c, v, c_a, r_a, c_f, r_f, to be measured. But the precision discussion will show that this number can, in practice, be considerably reduced.

The foregoing formula applies to a motor. A generator would be treated in precisely the same way; that is, it would be run as a motor under no load and c, v, etc., measured. Then to obtain an expression for its efficiency as a generator at any specified rate of output C and V we should have merely to substitute the values of c, v, C, V, etc., in the expression [98] for E. For the purposes of the precision discussion, however, we should not employ that expression, but should simplify it thus:

$$E = \frac{O}{O+L} = \frac{O-L}{O} \text{ approx.} = 1 - \frac{L}{O} \text{ approx.} \quad [106]$$

This expression could not properly be used to compute E, but is exact enough for the precision discussion. Thus we should have for the generator

$$E = 1 - \frac{L}{O} = 1 - \frac{1}{CV}\left\{ C_a^2 R_a + C_f^2 R_f + \frac{S}{s}(cv - c_a^2 r_a - c_f^2 r_f) \right\}, \quad [107]$$

which is identical in form with that for the motor.

In either case the input or output corresponding to CV may be anything we choose, e.g., half load, three-quarters load, full load, etc. This will be more fully perceived in the example.

Problem.—A shunt-wound motor is to be tested for commercial efficiency e by the Stray-power Method. It is rated at 220 volts, 40 amperes, and 1200 revolutions per minute, and is stated by the maker to have an armature resistance of 0.14 ohm and a field resistance of 130 ohms. The precision desired in the value of e for full rated load is $\Delta e/e = 0.25$ per cent. Required to find by the precision discussion:

(*a*) Precision necessary in the measured components.

(*b*) Whether any of the components can be wholly omitted.

(*c*) How closely to their normal running temperature the field coils and the armature must be when measured.

(*d*) Whether any of the results of (*a*) could be applied to other motors; and if so, under what conditions.

Solution.—For the solution we require approximate values of c, v, r_a, r_f, R_a, and R_f. The better way would be to make a preliminary run and measure these quantities roughly. But it is usually more convenient to make the discussion in advance of any trial. We may do so in this case as follows: Assume that $r_a = R_a = 0.14$ ohms, also that $r_f = R_f = 130.$ ohms, the values stated by the maker. Both of these assumptions will be proved to be close enough by the results of the discussion. We are obliged further to assume a value of e in order to deduce a value for c. This we can usually also do closely enough for the preliminary discussion from inspection of the machine. Suppose that in the present case we estimate the efficiency to be about 88. per cent at full load. Then if $v = 220.$, c must be $0.12 \times 40. = 4.8$ amp., since $100 - 88 = 12$ per cent of the power and therefore of the current applied under normal voltage.

The expression above deduced for e must be slightly modified to meet this case, for c_a and c_f are not here measured, but

$$c_f = \frac{v}{r_f}, \quad \text{and} \quad c_a = c - \frac{v}{r_f}.$$

The expression for e, then, containing all the components to be measured properly expressed for the precision discussion is

$$e = 1 - \frac{1}{CV}\left\{\left(C - \frac{V}{R_f}\right)^2 R_a + \frac{V^2}{R_f} + \frac{S}{s}\left[cv - \left(c - \frac{v}{r_f}\right)^2 r_a - \frac{v^2}{r_f}\right]\right\}. \quad [108]$$

The components to be measured in the test are thus seven, viz., R_a, R_f, s, c, v, r_a, and r_f, so that $n = 7$. We have therefore to apply the general formula [37] to these. The following simplification may be made. As s may easily be made within a few per cent of S in the run we may regard S/s as unity in all differentiations except that with respect to s. The values of C and V are not measured values, but simply stated to define the condition at which the efficiency is to be computed. Not being measured they are not subject to errors of measurement,

and are therefore not to be differentiated. Proceeding then with the differentiation and substituting numerical values gives

$$\frac{de}{dR_a} = -\frac{1}{CV}\left\{\left(C-\frac{V}{R_f}\right)^2\right\} = -\frac{1}{6.3},$$

$$\frac{de}{dR_f} = -\frac{1}{CV}\left\{2\left(C-\frac{V}{R_f}\right)\frac{VR_a}{R_f^2}-\frac{V^2}{R_f^2}\right\} = +\frac{1}{3.1\times 10^3},$$

$$\frac{de}{ds} = +\frac{1}{CV}\left\{\frac{S}{s^2}\left[cv-\left(c-\frac{v}{r_f}\right)^2 r_a-\frac{v^2}{r_f}\right]\right\} = +\frac{1}{1.5\times 10^4},$$

$$\frac{de}{dc} = -\frac{1}{CV}\left\{v-2\left(c-\frac{v}{r_f}\right)r_a\right\} = -\frac{1}{4.0\times 10^1},$$

$$\frac{de}{dv} = -\frac{1}{CV}\left\{c+2\left(c-\frac{v}{r_f}\right)\frac{r_a}{r_f}-\frac{2v}{r_f}\right\} = -\frac{1}{6.3\times 10^3},$$

$$\frac{de}{dr_a} = +\frac{1}{CV}\left\{\left(c-\frac{v}{r_f}\right)^2\right\} = -\frac{1}{9.1\times 10^2},$$

$$\frac{de}{dr_f} = -\frac{1}{CV}\left\{-2\left(c-\frac{v}{r_f}\right)\frac{vr_a}{r_f^2}+\frac{v^2}{r_f^2}\right\} = -\frac{1}{3.0\times 10^3},$$

To find the numerical values of δR_a, etc., by [37], we must have Δe. Now $\Delta e/e = 0.0025$ and $e = 0.88$; $\therefore \Delta e = 0.0025 \times 0.88 = 0.0022$. Hence $\Delta e/\sqrt{n} = 0.0022/\sqrt{7} = 0.0022/2.7 = 0.00081$; $\therefore$

(*a*)	$\delta R_a = -$	8. $\times 10^{-4} \times$	6.3	$= -$ 0.0050	ohms	$= -$ 3.6 %
	$\delta R_f = +$	"	$\times$ 3.1 $\times 10^3$	$= +$ 2.5	ohms	$= +$ 1.9 %
	$\delta s = +$	"	$\times$ 1.5 $\times 10^4$	$= +$ 12.	r. p. m.	$= +$ 1.0 %
	$\delta c = -$	"	$\times$ 4.0 $\times 10^1$	$= -$ 0.032	amperes	$= -$ 0.67 %
	$\delta v = -$	"	$\times$ 6.3 $\times 10^3$	$= -$ 5.0	volts	$= -$ 2.3 %
	$\delta r_a = +$	"	$\times$ 9.1 $\times 10^2$	$= +$ 0.73	ohms	$= +$ 500. %
	$\delta r_f = -$	"	$\times$ 3.0 $\times 10^3$	$= -$ 2.4	ohms	$= -$ 1.8 %

(*b*) (*c*) Inspecting these values the following points may be noted. (1) The armature resistance, r_a, in the run under no load is entirely negligible. A single measurement, viz., R_a, with the armature at the temperature of full load is all that is needed; hence we might use $n = 6$ instead of $n = 7$. (2) The

armature coils being of copper will change in resistance by 0.4 per cent per degree centigrade. The maximum allowable change is 3.6 per cent, which corresponds to $3.6/0.4 = 9.°$ C. Some care is therefore essential that the armature is fully warmed up to its normal state when R_a is measured. (3) The values of δR_f and δr_f are nearly equal and of opposite sign, and obviously $R_f = r_f$ very nearly. Hence if the same numerical value be used for both, any error in that value will be nearly eliminated, so nearly in fact that a large error will be admissible. We may see how large by substituting r_f for R_f, and then differentiating e with respect to r_f. This gives

$$\frac{de}{dr_f} = -\frac{1}{CV}\left\{2\left(C - \frac{V}{r_f}\right)\frac{VR_a}{r_f^2} - \frac{V^2}{r_f^2} + \left[-2\left(c - \frac{v}{r_f}\right)\frac{vr_a}{r_f^2} + \frac{v^2}{r_f^2}\right]\right\} = -\frac{1}{6.8 \times 10^4},$$

$$\therefore \delta' r_f = -8 \times 10^{-4} \times 6.8 \times 10^4 = -54. \text{ ohms} = -42.\ \%.$$

This clearly shows that one rough measurement of the field resistance is sufficient. The temperature may be $42./0.4 = 105°$ C. from normal, that is, the cold resistance would be near enough. We might even use the stated resistance without measurement. Hence also the error from this source can be easily rendered of negligible amount, and we may use $n = 5$ instead of $n = 7$ as above, so that $\Delta e/\sqrt{n}$ would become $0.0022/2.2 = 0.0010$ if the discussion were to be revised. The values admissible for δv, δc, etc., would then be correspondingly increased. The three sections of this paragraph answer questions (*b*) and (*c*) of the problem.

(*d*) The results above obtained could be applied without material error to any motor not differing from this one by more than about 10 per cent in the rated values of V, C, R_a, and S, or by more than 40 per cent in R_f.

Note that if the efficiency for other than full load is to be found with the same or any given precision, a solution similar to the above must be made by substituting the corresponding values of C and R_a for the desired load, e.g., 20 amp. (or more exactly 22 amp.) and about 0.14 ohms for half load.

Example XXXIII.—Cradle Dynamometer.—The principle and operation of the Brackett cradle dynamometer are so well known that a brief description is sufficient. The cradle consists of a platform suspended at each end from a horizontal knife-edge, both being in the same line. The dynamo (generator or motor) is placed securely upon the platform and adjusted until the axis of rotation of its shaft is coincident with the line of the knife-edges. The centre of gravity of the whole system is then raised or lowered by means of auxiliary weights until just below the axis. The system then oscillates slowly and sensitively like a balance. When the dynamo is in operation the mechanical power applied to or given out by its pulley tends, through the magnetic reaction between the armature and fields, to rotate the machine and therefore the whole system. This tendency is counterbalanced by a weight hung upon a horizontal lever arm projecting from the dynamometer at right angles to the axis. The weight or its distance or both are adjusted until a balance is obtained when this weight w into its horizontal distance l from the axis gives the rotary moment of the system, and therefore that applied to or given out by the dynamo. From this and the speed of the dynamo the power can be computed.

Problem.—The commercial efficiency at full load of a certain generator is to be measured by a cradle dynamometer. The dynamo is rated at 75 volts, 60 amperes, and 1400 rev. per min., and it has probably an efficiency of about 90 per cent. The diameter of its pulley is $2R = 10$ inches. The dynamo and dynamometer together weigh about 3000 lbs. The length of the arm to carry the dynamometer weights is $l = 3.4$ ft. Required in advance of the test a precision discussion of the proposed measurement and of the dynamometer. The value of E is desired to one per cent.

Solution.—First. Precision necessary in each measured component. The expression for the efficiency in terms of the measured components is

$$E = \frac{O}{I} = \frac{CV}{746} \cdot \frac{33\,000}{2\pi l n w} \cdot \quad . \quad . \quad . \quad . \quad [109]$$

The measured quantities are C, V, l, n, and w; $\therefore n = 5$. By [53] we have for equal effects

$$\frac{\delta C}{C} = \frac{\delta V}{V} = -\frac{\delta l}{l} = -\frac{\delta n}{n} = -\frac{\delta w}{w}$$

$$= \frac{1}{\sqrt{n}} \cdot \frac{\Delta E}{E} = \frac{1}{\sqrt{5}} \times 0.01 = 0.0045.$$

Each of these components then must be measured to about 0.45 per cent.

By the rules for significant figures the constant π must be carried to 5 places, i.e., 3.1416 must be used. This is, however, in excess of the strict requirement in this case, as will almost always be true when those rules are applied, since they are framed to cover the worst possible case. Applying the criterion for constants p. 70, we have, to be negligible,

$$\frac{\delta \pi}{\pi} \leqq \frac{1}{3} \frac{\Delta E}{E} \leqq 0.0015; \quad \therefore \delta \pi = 0.0047.$$

Hence 3.14 would in this case be close enough. In the computation of E evidently, by the rules, 5 places must be retained in each component factor, so that retaining $\pi = 3.1416$ does not materially increase the labor of computation. The constant 746 (see next page) varies with the force of gravitation, and therefore must be corrected for latitude and elevation. By the criterion for constants it must be exact to

$$\frac{\delta(746)}{(746)} \leqq 0.0015.$$

The change in g, and hence in the constant, is only about 0.003 between Edinburgh and the equator, and is only at the rate of about 0.01 per cent per 1000 ft. of elevation, so that these corrections are negligible in the present case. There is, however, a constant error of about 0.3 per cent in the legal ohm and volt of 1884, so that the constant 746 which is calculated for the theoretical volt is about 0.3 per cent too large if legal units are employed. For the present case this is barely worth

considering. It might, however, be worth while to employ $746\ (1 - 0.003) = 744.$ instead of 746.*

Second. Errors of Method, i.e., Errors and adjustments of the cradle dynamometer. It will be seen as we proceed that there are four of these ($\therefore p = 4$), and we wish to determine how small each error or how close each adjustment must be in order that the error shall be of negligible effect. By [80] the admissible limit will be such that the resulting error $\frac{\Delta_1 E}{E}$ in E shall be $\leqq \frac{1}{3} \cdot \frac{\Delta E}{E\sqrt{p}} \leqq 0.0017$. As the effect of these enters through the factor $I = \frac{2\pi lnw}{33\,000}$ which contains 3 of the 5 measured components, our discussion will incidentally give us the closeness of adjustment, etc., in the dynamometer necessary for measuring the mechanical power with it to $\sqrt{3/5} \times 0.01 = 0.77\%$, or about $\frac{3}{4}$ of one per cent, with the given load and power.

The adjustments and errors referred to are as follows: First, two, (a) and (b), which enter however rigid may be the construction of the dynamometer. Second, two, (c) and (d), which arise from yielding of the structure, i.e., from want of perfect rigidity of construction and of attachment of parts.

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* This constant 746 for reducing watts to h.p. is derived as follows:

$$1 \text{ lb.} = 13\,825g \text{ ergs.}$$

$$g = 980.6 - 2.5 \cos 2\lambda - 0.00\,000\,3h,$$

where units are c. g. s., λ = latitude, h = ht. above sea in cm. Thus for lat. 45° at sea-level $g = 980.6$ and

$$1 \text{ h.p.} = 550 \times 13\,825 \times 980.6 = 7.456 \times 10^9 \text{ ergs per sec.}$$

Now 1 volt-ampere or 1 watt by definition $= 10^8 \times 10^{-1} = 10^7$ ergs per sec.

$\therefore$ 1 *h.p.* = 745.6 watts at 45°, sea-level,

" = 745.5 " " Boston, sea-level,

" = 745.9 " " place where $g = 981$.

The latter value or 746 which is sensibly equal to it is ordinarily adopted. But it is important to remark as above shown that these values are about 0.3 per cent too large if the legal units of 1884 are used.

(*a*) *Zero or index error.* During the run it is impossible, of course, to maintain the dynamometer at its normal position of equilibrium, owing to continual slight fluctuations in power, to jarring, and to the swinging of the machine. This position is usually indicated by the position of a pointer or index carried by the lever-arm of the dynamometer. The reading of this index upon a scale is taken with the machine at rest, and during the run the index should be maintained at this point of rest. As this cannot be done exactly let n (in divisions of the scale) denote the average value of the index error. Then the dynamometer with dynamo in place must have sufficient sensitiveness so that the fractional error in the weight w at l due to this index error shall be negligible.

To find this limit let a weight p be found by trial which if put on at l will deflect the index by one division. Then the sensitiveness of the system is pl, i.e., this is, the rotary moment corresponding to one division. The fractional index error in the total measured moment wl found during the run will then be npl/wl. To be negligible this must be

$$\frac{npl}{wl} \leqq 0.0017\,; \quad \therefore p \leqq 0.0017\frac{w}{n}.$$

Suppose that inspection of the apparatus shows that n will be about 0.25 division. We require also the value of w, which may be found as follows. The output of the dynamo is $75 \times 60 = 4500.$ watts, which is $4500/746 = 6.0$ h.p. The efficiency of the generator being about 90 per cent, the mechanical power will be $6.0/0.9 = 6.7$ h.p.

$$\therefore \frac{2\pi lnw}{33\,000} = 6.7\,; \quad \therefore w = \frac{6.7 \times 33\,000.}{2 \times 3.1 \times 3.4 \times 1400.} = 7.5 \text{ lbs.}$$

Thus the sensitiveness in order that the index error may be negligible must be

$$p \leqq 0.0017\frac{7.5}{0.25} \leqq 0.050 \text{ lb.} = 0.8 \text{ oz.}$$

This sensitiveness may usually be easily reached or exceeded.

(*b*) *Shaft to line of knife-edges.* The axis of the shaft must lie strictly in the line joining the knife-edges supporting the cradle. Otherwise the resultant pull of the belt on the dynamo pulley, since it acts through and at right angles to this axis, will tend to rotate the dynamometer. Thus if the axis is displaced parallel to the knife-edge line by a distance d, and if $2a$ is the sum of the tensions in the two sides of the belts assumed parallel, then the resultant belt pull is $2a$ and the moment of rotation which this exerts upon the dynamometer is

$$2ad \sin \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [110]$$

where θ is the angle between the directions of d and $2a$. If the index is adjusted to zero, or its position of rest taken, with the belt on and tight, this erroneous moment would be counterbalanced once for all by the adjusting weights of the dynamometer, provided that $2a$, i.e., the belt tension, did not change during the run. This constancy is, however, hopeless. The tension will change sensibly between running and rest, and more or less progressive change of length and consequently of tension will occur during the run. Let us then first see how small d must be in order that a change equal to the whole belt pull $2a$ (i.e., one due to throwing on and off the belt) shall be of negligible effect. In the present case with a leather belt on an iron pulley we may assume, as shown below,* $2a = 360$. lbs. Further, as θ may have any value, let us solve for the average value $\frac{2}{\pi} = 0.65$ which $\sin \theta$ would have in the long run

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*The value of $2a$ may be calculated as follows: Let t = tension in tight side of running belt, and s = that in slack side, and a = one half the sum of these. Then

$$\frac{t}{s} = \frac{a + \frac{1}{2}(t-s)}{a - \frac{1}{2}(t-s)}.$$

For a leather belt on an iron pulley t/s cannot be more than about $\frac{5}{2}$ without undue slip, and on tight dynamo belts it is likely to be much less than this. Equating to $\frac{5}{2}$ and solving gives $a = 2(t-s)$. A value of $a = 3(t-s)$ would be of more frequent occurrence.

In this problem we have obviously

if all values of θ were equally probable. The fractional error in I and therefore in E due to this cause is then

$$2ad \sin \theta/wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad [111]$$

which to be negligible must be $\leqq 0.0017$. Whence to be of negligible effect d must be

$$d \leqq 0.0017 \frac{wl}{2a \sin \theta} = 0.0017 \frac{7.5 \times 3.4}{360 \times 0.65} = 0.00019 \text{ ft.}$$

$$= 0.0023 \text{ inch.}$$

It is obvious that no such adjustment as this can be made, and therefore the index *must* be adjusted to zero or the position of rest taken with the belt on and tight. If this is done, then the error from the steady pull will be eliminated, and all rapidly oscillating changes will obviously merely cause oscillations of the dynamometer and will eliminate themselves. But progressive changes will cause error. How much such change will occur? The answer must be largely a matter of estimate. Let us assume that a change of one tenth of $2a$, i.e., of 36 lbs., is as large as we need expect. Then for this to be of negligible effect, we must have

$$d \leqq 0.0017 \frac{7.5 \times 3.4}{36 \times 0.65} = 0.0019 \text{ ft.} = 0.023 \text{ inch.}$$

As close an adjustment as this is hardly to be expected on such an apparatus. Hence this source of error cannot probably be rendered negligible, and its effect on E may even exceed some of the other errors of measurement.

$(t - s)R = wl$; $\therefore t - s = \frac{7.5 \times 3.4}{1\frac{5}{8}} = 60.$ lbs.

$\therefore a = 180.$ lbs., and $2a = 360.$ lbs.

(*c*) The pull of the belt will also cause error in two other ways. As neither the dynamometer nor dynamo can be made perfectly rigid and inflexible, and as the dynamo cannot be attached to the dynamometer with perfect firmness, the pull of the belt will strain the structure somewhat. If the shaft of the dynamo be aligned to the knife-edges with the belt off, then when the belt is thrown on, this adjustment will be disturbed owing to the yielding of the structure. Both the axis of rotation of the shaft and the centre of gravity of the whole system will be thrown out of adjustment thereby. There will thus be introduced two erroneous moments, one of the same character as (*b*), that is due to the resultant belt-pull acting through a point outside of the line of the knife-edges; the other due to the weight of the system acting through the displaced centre of gravity. The first of these will be now discussed, the second under the heading (*d*).

How small must the displacement d of the shaft by the belt-pull be, in order that the error due to the pull shall be negligible? Obviously the solution will be precisely the same as in (*b*), except with respect to the value of θ. The displacement of the axis in this case will not be equally likely to occur in any direction, neither will it be always or in general in the direction of the belt-pull. The direction will be determined by the constraint of the various parts of the structure, but will tend to be most largely in the direction of the pull. No general average value of sin θ can then be stated, but the average would be less than the $\frac{2}{\pi}$ employed in (*b*). We shall therefore obtain an excessive but safe limit for d by using $\frac{2}{\pi}$ as before. The erroneous moment if the shaft is displaced by d is then

$$2ad \sin \theta,$$

and to be negligible the value of d must then as before be

$$d \leqq 0.0023 \text{ inch.}$$

This amount of displacement must then not occur with the total belt-pull if the zero adjustment is made only with the belt off, or must not occur with one tenth of $2a$ if the zero adjustment is made with the belt on, and we again accept that as a limiting value of the progressive change in belt-pull. It is doubtful whether the rigidity called for by this limit can be reached, and whether this error will not enter with at least the same magnitude as that under (*b*).

(*d*) Let h denote the horizontal component of the displacement of the centre of gravity. Then the weight W of the whole system acting at right angles to this will cause an erroneous moment tending to rotate the dynamometer whose amount will be Wh, which must be counterpoised on the lever arm. The fractional error in the power measurement will therefore be

$$\frac{Wh}{wl},$$

which to be negligible must be $\leqq 0.0017$. Hence for negligibility

$$h \leqq 0.0017\frac{wl}{W} = 0.0017\frac{7.5 \times 3.4}{3000} = 0.00\ 0015 \text{ ft.} = 0.00\ 018 \text{ in.}$$

This amount of horizontal displacement must then not be produced by whatever progressive change of belt-pull may occur (e.g., one tenth of $2a = 36$ lbs. as above), or by the total belt-pull if the zero point is not taken with the belt on. Such rigidity is not to be hoped for with a horizontal belt-pull or with the belt in any direction but the vertical. The belt must therefore be vertical and should preferably run downward, as this tends to least distortion of the structure. Just how much the centre of gravity would be displaced by any given pull with any given apparatus is hardly determinable. The only thing to be done is to have the dynamometer designed for great rigidity and lightness, and to see that the dynamo and all attachments are well secured. An important point in the design is the stiffness of the upright screws which carry the counterpoise blocks above the axis of rotation.

The uncertainty introduced by this source of error (d) is probably the greatest entering into the use of the dynamometer.

Summary.—From the foregoing considerations then we see that

Dynamo shaft must be adjusted to line of knife-edges with utmost care.

Zero reading or adjustment must be made with belt on and tight.

Dynamometer must be of extremely rigid design, and as light as possible; and all attachments must be firm.

Belt must run vertically downward from the dynamo.

Belt-pull must be as slight as possible, and therefore belt must be slack and pulley of large diameter. It would be better to run the dynamo by a couple without thrust.

When all these are attended to it is doubtful whether a measurement of power accurate to 1 per cent can be obtained.

Example XXXIV. — Tangent Galvanometer. — Problem. Given a good primary tangent galvanometer of the type described below; required a preliminary discussion to show what precision can be obtained in its use.

Description.—Coil of n turns having a mean radius r of about 20 cm., and being of rectangular section of breadth $2b = 2.0$ cm., and depth $2d = 2.4$ cm., about. Needle, a bundle of bits of watch-spring the distance between poles being $2l = 1$ cm., approximately, suspended by a single silk fibre, the coefficient of torsion being θ. Index attached to the needle and consisting of a bit of spun black glass. This moves over a graduated circle of about 10 cm. diameter, divided into degrees and secured to a circular mirror to reduce parallax. Readings taken to $0°.1$ by eye estimation, both ends of index and reversals of current being read, making four readings to be averaged. The expression for the current corresponding to an observed mean deflection $\phi°$ is

$$C = \frac{10H}{G} \cdot \tan \phi \cdot ( ) \cdot ( ), \text{ etc.}, \quad . \quad . \quad . \quad [112]$$

where H is the horizontal component of the intensity of the earth's magnetic field at the needle, G is the constant of the galvanometer, and the several ( ) represent correction factors for coil section, length of needle, torsion, etc.

Solution.—The statement of the problem assigns no definite value for the precision to be attained, but the requirement is to ascertain what precision may be obtained. As convenient a way as any of attacking this problem is to solve for the value of $\delta H/H$, which will produce separately an error $\delta C/C$, of some suitable amount, e.g., 0.001, and to make a similar solution for the same value of $\delta C/C$ for each measured component and for the various corrections, adjustments, etc. These results can then be conveniently discussed to ascertain whether they can be attained or exceeded, and a final summing up then made to see what resultant precision $\Delta C/C$ is attainable.

We have first to prepare the expression for C for discussion. As G is calculated from the measured dimensions of the coil, it must be expressed in terms of them. Suppose the radius of the coil to be found by measuring its inner and outer circumference at the time of winding. It will be exact enough to regard this as one measurement of its circumference s. Then

$$G = \frac{2\pi n}{r} = \frac{2\pi n}{\frac{s}{2\pi}} = \frac{4\pi^2 n}{s}. \quad . \quad . \quad . \quad . \quad [113]$$

The quantity H is the result of a complex indirect measurement. As we do not care to complicate the discussion of the present problem by introducing the detailed discussion of H, we will treat it as a directly measured component and leave its further consideration for separate treatment. The expression then in proper form is

$$C = \frac{10Hs}{4\pi^2 n} \cdot \tan \phi \cdot ( ) \cdot ( ), \text{ etc.} \quad . \quad . \quad . \quad [114]$$

The several parentheses will be written in full below. The simplest method of treatment is by separation into factors which are functions of one or more components, as in [75]. These will be denoted by f_1, f_2, etc. They are $f_1 = H$, $f_2 = s$, $f_3 = n^{-1}$, $f_4 = \tan \phi$, $f_5 = ( )$, etc.

$f_1 = H$.—By the conditions of the problem then

$$\frac{\delta H}{H} = \frac{\delta C}{C} = 0.001.$$

The measurement of H with this accuracy is difficult and laborious; moreover, the diurnal and local fluctuations are of about this order of magnitude, though the latter may be much larger. It is hardly practicable in any ordinary laboratory work to depend upon H as constant within about twice this limit, or 0.2 per cent, and this only under favorable conditions. By a good magnetometer it may be measured to less than 0.2 per cent. In merely relative measurements of C the absolute value of H need not be known and the effect only of variations in H enters.

$f_2 = s$.—We must have $\frac{\delta s}{s} = \frac{\delta C}{C} = 0.001$. As $r = 20$ cm., $s = 2\pi r = 120.$ cm. $\therefore \delta s = 0.001 s = 0.12$ cm. The error in r involves not only errors in the measurement of s, but irregularity in the distribution of the convolutions of the coil. The measurement of s can doubtless be made closer than 0.12 cm., but the uncertainty with respect to irregular winding owing to varying tension, to varying thickness of insulation, etc., will probably not be much less than that amount. We may probably count on this limit as about what is practically attainable in a good coil carefully wound into a channel.

If the coil is not a true circle but is more or less elliptical, the expression above given for G will not be exact. To find the amount of error for a given ellipticity it would be necessary to deduce an expression for the field at the centre of an elliptical circuit which cannot readily be done. It is easy to see, however, that the field does not change materially for slight

eccentricity, for if a circular circuit be gradually deformed into an ellipse the flattened sides approach the centre at first at sensibly the same rate as that at which the bulging ends recede.

$f_3 = n$.—This is necessarily a whole number and not subject to error except through mistake in counting. A mistake of one turn in a thousand would correspond to the assigned limit of $\delta C/C$, but as the value of n will seldom be as large as 1000, no mistake is allowable.

$f_4 = \tan \phi$.—We will make the solution for $\phi = 45°$, but the result will apply without sensible error to any deflection between 30° and 60° as shown later.

$$\frac{\delta \tan \phi}{\tan \phi} = \frac{\delta C}{C} = 0.001 ; \quad \tan 45° = 1.0 ;$$

$$\therefore \delta \tan \phi = 0.001$$

From this we have to determine the corresponding value of $\delta\phi$ by [34].

$$\delta\phi = \delta \tan \phi \Big/ \frac{d \tan \phi}{d\phi} ;$$

$$\frac{d \tan \phi}{d\phi} = \sec^2 \phi = \frac{1}{\cos^2 \phi} ;$$

$$\therefore \delta\phi = 0.001 \cos^2 \phi = 0.001 \times 0.50$$
$$= 0.00\,050 \text{ in radian measure.}$$

$$\therefore \delta\phi = 0°.03.$$

The value of ϕ is a mean of four readings in which the tenths of a degree are estimated by the eye. These estimations if properly made will always give the nearest tenth. The extreme error of estimation will therefore be $+ 0°.05$ and $- 0°.05$, and the error will be equally likely to have any value between these limits. The average error of a single estimation will be (page 21) $0°.025$, and of the mean of four will be $0.025/\sqrt{4} = 0°.013$, which is negligible compared with the value

0°.03 above deduced. But there are other sources of error in ϕ, viz., irregularities in graduation, eccentricity, parallax, torsion, coils out of vertical plane, coils out of magnetic meridian. Of these the first is partly eliminated by the reading at four different points on the circle, the second by reading both ends of the index, the third by bringing the eye when reading into a position where the index just covers its reflection in the mirror; the fourth, fifth, and sixth are corrected for by the corresponding correction terms which are separately discussed below. The residual errors from these four corrections are classed separately and therefore do not need to be regarded as augmenting $\delta\phi$. The errors from the first three sources may obviously be of any amount according to construction of instrument, but need not exceed the limit of 0°.03 which may be regarded as practically attainable.

$f_s = \left(1 + \frac{1}{2}\frac{b^2}{r^2} - \frac{1}{3}\frac{d^2}{r^2}\right)$.—This is the correction term for finite dimensions of the rectangular coil section, and is sufficiently close where the depth does not exceed one tenth of the radius. It is in reality a correction to the value of G. $2b$ = breadth, $2d$ = depth. The term involves r and therefore the measured components, but as the ( ) differs from unity by only one or two per cent at most it may be omitted in discussing δs, as was done above, and r may now be treated as a constant. This factor contains then two measured quantities $2b$ and $2d$. The value of $\delta f_s/f_s$ corresponding to equal effects for each must then be $0.0010\sqrt{2} = 0.0014$, corresponding to which we have to find the values of $\delta(2b)$ and $\delta(2d)$. As f_s is sensibly = 1 we have

$$\delta f_s = 0.0014 f_s = 0.0014.$$

By [45] and [34]

$$\delta(2b) = 2\delta b = 2\delta f_s \Big/ \frac{d(\ )}{db}$$

$$= 0.0028 \Big/ \frac{b}{r^2} = 0.0028 \Big/ \frac{1}{20^2} = 1.1 \text{ cm.}$$

It is obvious that this limit is needlessly large. A negligible amount would be one third of this, viz., 0.37 cm., and we can easily measure the depth much closer than this. If then the breadth measurement be made to 4 or better to 1 or 2 mm., its residual error will be negligible.

Similarly for the depth measurement

$$\delta(2d) = 2\delta d = 2\delta(\,) \Big/ \frac{d(\,)}{dd}$$

$$= 0.0028 \Big/ \frac{2d}{3r^2} = 1.6 \text{ cm.}$$

Therefore if the depth be measured to 6 mm., or better to 1 or 2 mm., the residual error will be entirely negligible.

The correction itself would be negligible when the coil section was such that $2b < 0.4$ cm. and $2d < 0.6$ cm.

Inspection of the form of the correction shows that if the coil section be so designed that

$$\frac{1}{2}\frac{b^2}{r^2} = \frac{1}{3}\frac{d^2}{r^2}$$

the correction will vanish. Solving gives

$$\frac{b^2}{d^2} = \frac{2}{3}; \quad \therefore \frac{2b}{2d} = \frac{5}{6} \text{ approx.}$$

If the coil be wound to these relative dimensions then the correction may be omitted. Obviously the dimensions must be adhered to within the limits $\delta(2b) = 4$ mm. and $\delta(2d) = 6$ mm. in this case, or within limits having the proportion of $\delta(2b) : \delta(2d) = 2 : 3$ in any case. These limits must be considered not merely as referring to the outside dimensions of the coils, but to the density of winding as well. If the winding is not in "square order," but is more dense in the depth than in the breadth of the coil, the correction by the above formula with respect to the breadth will be too great relatively

to the depth about in the proportion of the relative densities, and a corresponding allowance must be made.

$f_\phi = \left(1 - \frac{3}{4}\frac{l^2}{r^2} + \frac{15}{4}\frac{l^2}{r^2}\sin^2\phi\right)$.—The expression for G gives the field at the centre of the coil due to a unit (c. g. s.) of current in the coil. The needle has, however, a finite length $2l$, and its poles therefore lie in a field which when ϕ is nearly zero is slightly less intense than at the centre, and which increases with ϕ. The second term in the correction takes account of the first of the less intensity, the third term of the want of uniformity of the field. The centre of the needle is, of course, assumed to be at the centre of the coil. The expression is sufficiently exact when $2l$ is less than one tenth of the diameter of the coil.

Since the value of f_ϕ in an extreme case differs from unity by only one or two per cent, it may be omitted when r and ϕ are being discussed, as was done above. We have then only one component here, viz., $2l$. Then

$$\delta(2l) = 2\delta l = 2\delta(\,) \Big/ \frac{d(\,)}{dl}.$$

As $f_\phi = 1$ very nearly

$$\delta f_\phi = \frac{\delta f_\phi}{f_\phi} = 0.0010 \text{ approx.}$$

$$\frac{d(\,)}{dl} = -\frac{3}{2}\frac{l}{r^2} + \frac{15}{2}\frac{l}{r^2}\sin^2\phi.$$

Inspection shows that the value of f_ϕ is greatest for $\phi = 60°$ if the galvanometer is used (for reasons later stated) only between 30° and 60°. We will therefore solve for the worst case. Substituting gives

$$\delta(2l) = 0.0020 \Big/ \frac{1}{200} = 0.40 \text{ cm.}$$

Greater accuracy than this is easily attainable. 0.40/3 = 0.13 cm. would be negligible, and this can be reached. As the needle can never be made as short as 0.13 cm. on account of torsion, the correction itself at 60° can never be negligible. But the length can be measured accurately enough so that the residual error shall be negligible. It should be noted that $2l$ the pole distance is about 0.85 of the total length of the needle if this is a thin rectangular prism.

The correction obviously vanishes when

$$\frac{3}{4}\frac{l^2}{r^2} = \frac{15}{4}\frac{l^2}{r^2}\sin^2\phi,$$

which solved for ϕ gives $\phi = 26°.6$. The correction is negative below and positive above that angle. As far as concerns this error alone, 26.° would then be a favorable angle at which to use the instrument, but there are other considerations which outweigh this, as will be presently shown. At 45° the correction term for the case in hand would be 1.0014 which is four times the negligible amount.

$f_1 = \left(1 + \frac{3}{2}\frac{x^2 + y^2 - 2z^2}{r^2}\right)$.—The expression for f_6 is based on the assumption that the centre of the needle is at the coil centre. This adjustment, especially with a suspended needle, cannot be made with exactness, and it is necessary to know how closely it must be made. The above expression gives the correction factor to be applied when the needle is slightly out of centre. x is the horizontal component and y the vertical component of the displacement of the needle in the plane of the coil, and z is the displacement along the axis of the coil. In other words, relatively to the centre of the coil the coördinates of the centre of the magnet in its displaced position are: z along the axis of the coil, and x and y at right angles to this and to each other. As a basis for numerical solution let us assume that we can always set the needle so near the centre that neither x, y, nor z shall exceed a given distance a more than

once in a thousand times. The correction will be a maximum when $x = y = a$ and $z = 0$, or when $z = y = 0$ and $z = a$. Let us assume further that the law of accidental placing of the needle is such that the magnitude of the correction terms will follow the general law of distribution of deviations. The maximum value will then be $3\frac{a^2}{r^2}$ and the average value $\frac{3}{4}\frac{a^2}{r^2}$. This assumption is surely not exact, but is probably sufficiently correct for the purpose. What value then must a have to render this correction negligible; for obviously we cannot well measure x, y, and z and correct for it each time we use the instrument.

We have then

$$f_7 = \left(1 \pm \frac{3}{4}\frac{a^2}{r^2}\right),$$

and in order that this shall be negligible we must have

$$\frac{3}{4}\frac{a^2}{r^2} \leq \frac{1}{3} \times 0.001.$$

For f_7 must not exceed $1 \pm \frac{1}{3} \times 0.001$.

$$\therefore \quad a^2 = \tfrac{4}{9} \times 0.001 \times 400 = 0.18 \text{ cm.};$$

$$a = 0.42 \text{ cm.}$$

Hence the centering will be close enough if x, y, and z never exceed 0.42 cm. This is easily possible, but requires care, and should always be attended to in setting up the instrument.

For the same reason as in f_5 and f_6, r is here treated as a constant.

$f_8 = \left(1 + \frac{\phi\theta}{\sin\phi}\right)$.—This is the correction factor for torsion when this is not unduly great; ϕ is to be expressed in degrees. It does not, however, take into account the effect of initial torsion, which is eliminated by the process of reversal of the current. Here θ is the "coefficient of torsion." If, as is

usual, θ is determined by reading the change α in the zero reading produced by twisting the fibre top (or bottom) through 360°, we have

$$\theta = \frac{\sin \alpha}{360 - \alpha}, \text{ or sensibly } \frac{\sin \alpha}{360},$$

and this should be substituted in the above expression to prepare it for discussion, since α is the measured quantity. Hence

$$f_s = \left(1 + \frac{\phi}{\sin \phi} \cdot \frac{\sin \alpha}{360}\right). \quad . \quad . \quad . \quad . \quad [115]$$

As in discussing f_s the deflection ϕ will be taken at 60°, since the correction then has its largest value.

How closely must α be measured?

$$\delta\alpha = \delta f_s \Big/ \frac{d(\,)}{d\alpha};$$

$$\frac{d(\,)}{d\alpha} = -\frac{\phi}{\sin \phi} \cdot \frac{\cos \alpha}{360} = -\frac{\cos \alpha}{5.2}.$$

A not unusual value for α is 3°, although it may easily be made less. For $\alpha = 3°$,

$$\delta\alpha = 0.0010/0.19 = 0.0052 \text{ rad., or } 0°.30.$$

This would correspond to $\delta C/C = 0.001$, but one third of it, viz., 0°.10 would be negligible, and as this is easily reached, the torsion can easily be corrected so that the residual error shall be negligible. In order that the entire correction may be omitted, we must have $\alpha \leqq 0°.10$, which can be attained, but requires an exceptionally fine fibre and strong magnetization of the needle. Inasmuch as the correction for length of needle cannot be rendered negligible, it would be better to use a longer needle, say 1.5 to 2.0 cm., thus making the torsion

negligible and throwing all the correction into the factor for length of the needle.

$f_9 = \frac{1}{\cos \beta}$.—The plane of the coils should be vertical, as that is the supposition upon which the law of tangents is deduced. If the coil is inclined through an angle β, we know by the same demonstration as for the cosine galvanometer that

$$C = \frac{10H}{G} \cdot \frac{\tan \phi}{\cos \beta}. \quad \ldots\ldots \quad [116]$$

Let us inquire what value of β would produce a negligible error, as we merely wish for a guide to show how accurately we must level the instrument. Perhaps the simplest method of solving is as follows. As β is small we may write $\cos \beta = 1 - x$, where x is a small fraction. Hence

$$\frac{1}{\cos \beta} = \frac{1}{1-x} = 1 + x \text{ approximately.}$$

To be negligible, we must have

$$x \leqq 0.00\,033\,; \quad \therefore \cos \beta = \geqq 0.99\,967\,; \quad \therefore \beta \leqq 1^\circ.5.$$

This is easily reached in levelling by plumb-line or otherwise, as it corresponds to a displacement of the top of the coil beyond the bottom by

$$2r \sin 1^\circ.5 = 1.0 \text{ cm.},$$

which is easily perceptible.

f_{10}.—When the galvanometer is to be used, the plane of its coils should be adjusted into the magnetic meridian. This is done either by finding the meridian by means of an auxiliary compass and setting the coil to correspond, or by turning the coil about a vertical axis until reversals of current show equal

deflections in opposite directions. The latter method is interfered with by any initial torsion which may be present in the fibre. Either method can, of course, only bring the coil more or less closely into the meridian, and it is essential to know how closely the adjustment ought to be made.

In Fig. 1 let NS show the direction of the magnetic meridian. This is the direction in which the axis of the undeflected needle will normally stand. Suppose the centre of the needle to be at P. And let QR represent the direction of the plane of the coils making an angle $OPQ = \omega$ with the meridian. Let PO represent the earth's horizontal component H, and PL the field F produced by a + current C through the coil. Then under this current the resultant field will be PB, and the needle will set in that direction, its deflection being $OPB = \phi_1$. With the same current reversed, the field will be equal and opposite to PF, and the resultant field will be PB', and the deflection $\phi_2 = OPB'$. In the use of the instrument the mean angle is employed, viz. $\frac{1}{2}(\phi_1 + \phi_2)$. The deflection ϕ_2 on the same side with ω is obviously greater than ϕ_1 on the opposite side. The use of the mean angle rests on the assumption that ϕ_2 is just as much greater as ϕ_1 is smaller than the true angle. This assumption is in general not exact, but is more nearly true as ω is smaller and as ϕ is more nearly 45°, being true for that angle whatever the value of ω, as will be shown later. It remains then to ascertain the algebraic relation between ϕ_1, ϕ_2, ω, and the true angle ϕ which would be obtained if ω were zero.

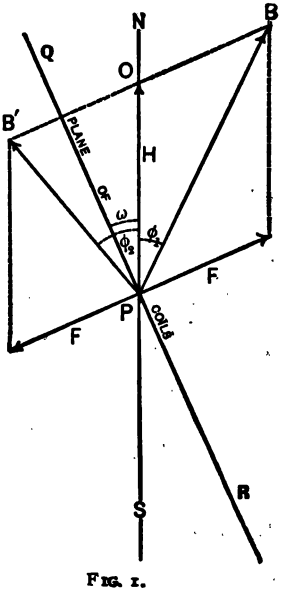


FIG. 1.

In Fig. 2 the letters correspond with Fig. 1. Suppose $\omega = 0$. Then the deflections would be OPA and OPA', and they

would both be the same, and each equal to the true angle ϕ. Suppose the coils to be inclined at some angle ω as in Fig. 1. Draw through O, Fig. 2, a line BOB' making any angle ω with AOA', and lay off along it $OB = OA = F$ and $OB' = OA = OA' = F$. Then obviously OB and OB' will be the fields due to the same current C with the coils inclined at $\omega°$; PB and PB' will be the resultant fields; and OPB will be ϕ_1 and OPB' will be ϕ_2. Prolong PB' to meet AA' prolonged in C'. PB cuts AA' at C. We proceed first to find expressions for tan ϕ_1 and tan ϕ_2.

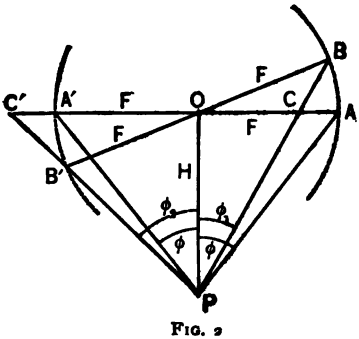


FIG. 2

In the triangle OCB we have

$$\frac{OC}{OB} = \frac{OC}{F} = \frac{\sin OBC}{\sin OCB} = \frac{\sin (90° + \omega + \phi_1)}{\sin (90° + \phi_1)} = \frac{\cos (\omega + \phi_1)}{\cos \phi_1} =$$

$$\tan \phi_1 = \frac{OC}{H} = \frac{F}{H} \cdot \frac{OC}{F} = \frac{F}{H} \frac{\cos (\omega + \phi_1)}{\cos \phi_1} =$$

$$\tan \phi \cdot \frac{\cos (\omega + \phi_1)}{\cos \phi_1} = \tan \phi \cdot \cos \omega - \tan \phi \cdot \sin \omega \cdot \tan \phi_1 ;$$

$$\therefore \tan \phi_1 = \tan \phi \cdot \cos \omega / (1 + \tan \phi \cdot \sin \omega). \quad . \quad [117]$$

In the triangle $OC'B'$ we have

$$\frac{OC'}{OB'} = \frac{OC'}{F} = \frac{\sin OB'C'}{\sin OC'B'} = \frac{\sin (90° - \omega + \phi_2)}{\sin (90° - \phi_2)} = \frac{\cos (\omega - \phi_2)}{\cos \phi_2}$$

$$\tan \phi_2 = \frac{OC'}{H} = \frac{F}{H} \cdot \frac{OC'}{F} = \frac{F}{H} \frac{\cos (\omega - \phi_2)}{\cos \phi_2} =$$

$$\tan \phi \cdot \frac{\cos (\omega - \phi_2)}{\cos \phi_2} = \tan \phi \cdot \cos \omega + \tan \phi \cdot \sin \omega \cdot \tan \phi_2 ;$$

$$\therefore \tan \phi_2 = \tan \phi \cdot \cos \omega / (1 - \tan \phi \cdot \sin \omega). \quad . \quad [118]$$

Hence

$$\tan(\phi_1+\phi_2)=\frac{\tan\phi_1+\tan\phi_2}{1-\tan\phi_1\cdot\tan\phi_2} \quad . \quad . \quad [119]$$

$$=\left\{\frac{\tan\phi\cdot\cos\omega}{1+\tan\phi\cdot\sin\omega}+\frac{\tan\phi\cdot\cos\omega}{1-\tan\phi\cdot\sin\omega}\right\}\Big/\left\{1-\frac{\tan^2\phi\cdot\cos^2\omega}{1-\tan^2\phi\cdot\sin^2\omega}\right\}$$

$$=\frac{2\tan\phi\cdot\cos\omega}{1-\tan^2\phi}=\frac{2\tan\frac{1}{2}(\phi_1+\phi_2)}{1-\tan^2\frac{1}{2}(\phi_1+\phi_2)}.$$

$$\therefore[1-\tan^2\tfrac{1}{2}(\phi_1+\phi_2)]\tan\phi\cdot\cos\omega=(1-\tan^2\phi)\tan\tfrac{1}{2}(\phi_1+\phi_2),$$

$$\tan\phi\cdot\cos\omega\cdot\tan^2\tfrac{1}{2}(\phi_1+\phi_2)+(1-\tan^2\phi)\tan\tfrac{1}{2}(\phi_1+\phi_2)=$$

$$\tan\phi\cdot\cos\omega;$$

$$\therefore\tan\tfrac{1}{2}(\phi_1+\phi_2)=$$

$$\frac{\tan^2\phi-1\pm\sqrt{\{\tan^4\phi-2\tan^2\phi+4\tan^2\phi\cdot\cos^2\omega+1\}}}{2\tan\phi\cdot\cos\omega}. \quad [120]$$

The upper sign is to be taken for the radical; for suppose that $\omega=0$, then

$$\tan\tfrac{1}{2}(\phi_1+\phi_2)=\frac{\tan^2\phi-1\pm(\tan^2\phi+1)}{2\tan\phi}$$

which is evidently correct with the upper sign only. Hence, as $2\cos^2\omega-1=\cos 2\omega$, we have finally

$$\tan\tfrac{1}{2}(\phi_1+\phi_2)$$

$$=\frac{\tan^2\phi-1+\sqrt{\{\tan^4\phi+2\tan^2\phi\cdot\cos 2\omega+1\}}}{2\tan\phi\cdot\cos\omega}, \quad . \quad [121]$$

as the desired expression connecting ϕ_1, ϕ_2, ω, and ϕ.

To find what value of ω is negligible, we may substitute successively values of $\omega=1°$, 2°, etc., with $\phi=30°$ and 60°, since the error increases with ϕ above and below 45°, being negative below 45°. We may plot these values and interpolate or may interpolate directly. That value of ω would be

negligible which would give such a value of $\tan \frac{1}{2}(\phi_1 + \phi_2)$ for $\phi = 60°$ that

$$\tan \tfrac{1}{2}(\phi_1 + \phi_2)/\tan \phi = 1 \pm 0.00\,033.$$

This will be found to be $\omega = 2°.1$. Thus if the plane of the coils is within about 2.° of the meridian the error will be negligible between $\phi = 30°$ and $\phi = 60°$. It is not difficult to adjust to this closeness.

By substituting $\phi = 45°$ in [120], less conveniently in [121], we have

$$\tan \tfrac{1}{2}(\phi_1 + \phi_2) = \frac{1 - 1 + \sqrt{\{1 - 2 + 4\cos^2 \omega + 1\}}}{2 \cos \omega} = \frac{2 \cos \omega}{2 \cos \omega} = 1,$$

but $\tan 45° = 1$. Hence at $\phi = 45°$ the mean angle $\frac{1}{2}(\phi_1 + \phi_2)$ is correct whatever the value of ω. That is, ϕ_1 is as much too large as ϕ_2 is too small, or *vice versa*. This rather surprising result is decidedly important as indicating that, so far as this source of error is concerned, 45° is the best angle by far to use in accurate work, since it eliminates wholly the error due to any imperfect adjustment into the meridian. The proof of this proposition may be much more easily arrived at graphically. Fig. 3 corresponds in all respects with Fig. 2 except that it is drawn for $\phi = 45°$, so that $F = H$. P therefore falls upon the circumference of a circle passing through A, A', B, B', etc. The angle $APA' = 2\phi$ is measured geometrically by half the semi-circumference ABA'. When $\omega = BOA$, the angle $BPB' = \phi_1 + \phi_2$, and is measured geometrically by half the arc $BDA'B'$, which is also a semi-circumference. Hence $\phi_1 + \phi_2 = 2\phi$. A similar statement is true for any other value of ω, thus proving the proposition.

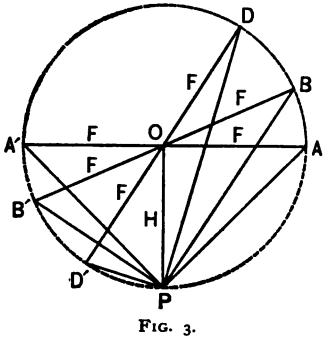


FIG. 3.

As above stated the most convenient way of adjusting the coils to the meridian is by sending the same current first in a positive and then in a negative direction through the coil, and

adjusting the coil until the deflections are equal. This is most readily done as follows. Set the galvanometer up approximately. Read the index with no current, calling this the zero reading. Send a current which deflects the needle by about 45°. Let the deflection corrected for zero be called ϕ_1. Reverse the current and let the corrected deflection be denoted by ϕ_2. Suppose ϕ_2 to be the greater, then

$$\omega = \phi_2 - \phi_1, \quad . \quad . \quad . \quad . \quad . \quad [122]$$

ω being always on the side of the largest deflection. To adjust, open the circuit and when the needle is at rest turn the coils through an angle $\omega = \phi_2 - \phi_1$ toward the side of the smallest deflection. The adjustment will then be very nearly right. It is best to take a new zero reading and repeat as a check or for closer adjustment. This method does not eliminate the effect of initial torsion of the fibre.

The proof is as follows. From the foregoing demonstration by substituting $\phi = 45°$, $\tan \phi = 1$, we have

$$\tan \phi_1 = \frac{\cos \omega}{1 + \sin \omega} = \cos \omega - \sin \omega, \text{ approx., as } \omega \text{ is small,}$$

$$\tan \phi_2 = \frac{\cos \omega}{1 - \sin \omega} = \cos \omega + \sin \omega, \text{ approx., as } \omega \text{ is small,}$$

$$\therefore \tan \phi_2 - \tan \phi_1 = 2 \sin \omega.$$

Let $\Delta = \phi_2 - \phi$. Then as $\phi = 45°$ we have by the above proposition of Fig. 3, $\phi - \phi_1 = \Delta$ also.

$$\therefore \phi_2 = \phi + \Delta, \quad \text{and} \quad \phi_1 = \phi - \Delta.$$

$$\therefore \tan (\phi + \Delta) - \tan (\phi - \Delta) = 2 \sin \omega$$

Now as Δ is small and $\tan \phi = 1$

$$\tan (\phi + \Delta) = \frac{\tan \phi + \tan \Delta}{1 - \tan \phi \cdot \tan \Delta} = \frac{1 + \Delta}{1 - \Delta} \text{ approx.} = 1 + 2\Delta \text{ app.}$$

$$\tan (\phi - \Delta) = \frac{\tan \phi - \tan \Delta}{1 + \tan \phi \cdot \tan \Delta} = \frac{1 - \Delta}{1 + \Delta} \text{ approx.} = 1 - 2\Delta \text{ app.}$$

$$\therefore \tan(\phi + \Delta) - \tan(\phi - \Delta) = 4\Delta = 2 \sin \omega = 2\omega \text{ approx.}$$

Now $\phi_2 - \phi_1 = 2\Delta$. $\therefore \omega = \phi_2 - \phi_1$, Q.E.D.

*f*11.—For inclination of mirror to horizontal. In the galvanometer here considered the graduated circle is supposed to be secured to the surface of the mirror and the mirror to be placed horizontally below the needle with its index. Since the index always covers its image when a reading is taken, the plane of sight, i.e., the plane containing the eye and the index, is always perpendicular to the mirror. But the index and needle necessarily deflect about a vertical axis. If, therefore, the mirror and circle be not horizontal but inclined, the angle read off upon it will not be the true angle ϕ swept through by the needle, but an oblique projection of that angle. It is, then, essential to know how nearly horizontal the mirror must be to avoid sensible error from this cause.

In Fig. 4 let the circle $I'A'J'A''$ show the graduated circle

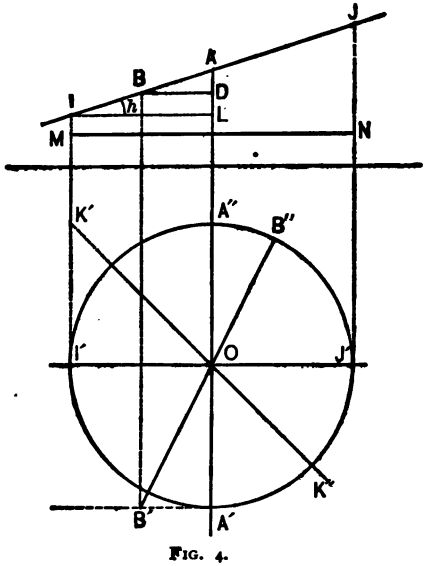


FIG. 4.

in horizontal projection and MN its vertical projection. Suppose the needle to be at A. Then if index and circle are both

horizontal, the circle would be as shown in Fig. 4, and the plane swept through by the index would be shown by a horizontal line through A. But if the mirror were inclined, then the plane of the index would still be horizontal, but that of the mirror would be inclined to it. The result as far as angular readings are concerned, however, would be the same as though the mirror remained horizontal and the plane swept through by the index were inclined. It is more convenient to represent the latter in the drawing. Therefore let IAJ represent the vertical projection of the plane of the index inclined to the mirror at an angle $AIL = h$.

First. Suppose the index when undeflected to stand in the direction $A'A''$. In vertical projection it will appear as a point at A. Let it be deflected through an angle whose true value is ϕ, but which is read on the circle as $\phi' = A'OB'$. What is the relation between ϕ and ϕ'? At A' draw a tangent $A'B'$, and prolong $B''OB'$ to intersect this at B'. Then

$$\tan \phi' = \frac{A'B'}{OA'}.$$

Project B' upward to B on IJ. Then

$$\tan \phi = \frac{AB}{OA'};$$

for AB is the true length of the tangent at A cut off by the pointer and shown in horizontal projection in $A'B'$, and OA' is shown in its true length. Therefore

$$\frac{\tan \phi}{\tan \phi'} = \frac{AB}{A'B'} = \frac{AB}{BD} = \frac{1}{\cos h}.$$

$$\therefore \tan \phi = \frac{1}{\cos h} \tan \phi'; \quad . \quad . \quad . \quad . \quad . \quad [123]$$

and the correction factor for the inclination of the mirror is, in this case, $1/\cos h$, which is the same as though the coils were inclined as in the cosine galvanometer.

Second. Suppose that the zero position of the index were $I'J', IJ$. Let the needle be deflected through any angle shown in projection upon the mirror as $\phi' = I'OK'$. Thus

$$\tan \phi' = \frac{I'K'}{I'O}.$$

As the tangent line of which $I'K'$ is the projection is parallel to the mirror it appears in this projection at its full length and in the vertical projection as a point at I. Then

$$\tan \phi = \frac{I'K'}{AI},$$

since AI is the true length of the side of the angle ϕ which appears as OF' in the mirror projection. Therefore

$$\frac{\tan \phi}{\tan \phi'} = \frac{I'O}{AI} = \frac{IL}{AI} = \cos h. \quad . \quad . \quad . \quad [124]$$

Hence the correction factor for the inclination of the mirror is in this case cos h.

The first of these cases is where the mirror is tipped about a horizontal line through the zero points; the second, where the tipping is about a line through the 90° points. For a tipping about any other line the effect could be ascertained by resolving it into two parts, one with reference to each of the above positions. The effect will be intermediate between the two above extremes, so that it need not be further considered.

What value of h will produce the limiting negligible error? As h is very small we may write $\cos h = 1 - x$, where x is a small fraction. Then for the first case

$$\frac{1}{\cos h} = \frac{1}{1 - x} = 1 + x \text{ approx.}$$

For the second case we have simply

$$\cos h = 1 - x.$$

As these enter as direct factors we must have for negligibility

$$x \leqq 0.00\,033.$$

$$\therefore \cos h \geqq 1 - x \geqq 0.99\,967;$$

$$\therefore \quad h \leqq 1^\circ.5.$$

This error can be rendered negligible without difficulty by due care in the original construction of the instrument and by proper levelling at the time of use. The most convenient method for the latter is to have a plumb-line hanging from a marked point, and arranged to be brought over a reference point on a plate attached to the lower part of the coil, pains being taken to see, once for all, that the mirror is level and the coil vertical when the line so indicates. It should be noted that neither reversal nor reading both ends of the needle tends to eliminate this error.

Summary.—The following table gives a summary of the results. By bringing the adjustments or by measuring the quan-

No.	For $\frac{\delta C}{C}$ =	Are Required:—	
1	0.002	$\delta H/H = 0.002$	Earth's field.
2	0.001	$\delta s/s = 0.001$	Circumference of coil.
3	0.000	δn = zero	Turns in coil.
4	0.001	$\delta\phi = 0^\circ.03$	Deflection.
5	Negligible	$\left\{ \begin{array}{l} \delta(2b) = 4.\text{ mm.}, \\ \delta(2d) = 6.\text{ mm.} \end{array} \right\}$	Coil section.
6	"	$\delta(2l) = 4.$ mm.	Length of needle.
7	"	$a = 4.$ mm.	Centering of needle.
8	"	$\delta\alpha = 0^\circ.10$	Torsion.
9	"	$\beta = 1^\circ.5$	Coil out of vertical.
10	"	$\omega = 2^\circ.0$	Coil out of meridian.
11	"	$h = 1^\circ.5$	Mirror out of horizontal.

tities designated within the limits given in the table, the residual errors from all the sources except the first four may be made negligible with reference to $\delta C/C = 0.001$ from each. If their residuals were all present at this maximum amount of 0.0003 each their resultant effect would be $0.00\,033\sqrt{7} = 00\,086$.

Apart from H, the only remaining sources of error are 2 and 4, whose resultant effect would be $0.001\sqrt{2} = 0.0014$, compared with which the above amount of 0.00 086 is not quite negligible although the actual amount probably would be. But as above given the total value of $\Delta C/C$ would be $0.00\,033^2 \times 7 + 0.001^2 \times 2 = 0.0017^2$. Hence we ought to be able with such an instrument to obtain relative measurements with an accuracy of about 0.2 per cent. And if H can be measured and relied upon to 0.2 per cent as above assumed, we ought to be able to obtain absolute measurements to $0.2\sqrt{2} = 0.28$ or say 0.3 per cent.

Best Range of Deflections.—It is commonly stated that the best deflection at which to use a tangent galvanometer of the kind above discussed is 45°. This statement is insufficient and not wholly exact. What we wish to know is, what angle ϕ or what range of deflections will give the greatest fractional precision in the current causing them, as further explained at page 110. To determine this we have to consider the nature of those errors which are a function of the deflection, or which otherwise affect $\delta C/C$. These are such as result from 1st, $\delta\phi$; 2d, length of needle; 3d, torsion; 4th, coil out of meridian. Of these the 2d and 3d can readily be so corrected as not to enter sensibly, but still the residual error from the 2d will be least at 26°.6, and that from the 3d will diminish as the angle is smaller. The error from the 4th source can be eliminated by some care, but is more difficult of removal than the two preceding. Its residual is least at 45°, thus pointing to that angle as the best. The value of $\delta\phi$ is constant as far as errors of eye estimation are concerned, and the other sources of error making it up follow the law of accidental distribution. We must therefore treat $\delta\phi$ as constant for all values of ϕ. The following table gives the values of $\delta C/C$ for $\delta\phi = 0°.03$ for each 5° from 0° to 90°. Inspection of these, or better of a plot made from them, shows that $\delta C/C$ is sensibly constant between $\phi = 30°$ and $\phi = 60°$ although the minimum is at $\phi = 45°$. The precision is, however, only about one-half less at 20° and 70°, but beyond those points it falls off rapidly. Hence as far as this source of

error is concerned, any deflection between 30° and 60° is equally good, and between 20° and 70° nearly as good; but below 20° and above 70° should not be used in careful work.

$\phi =$	$\frac{\delta C}{C} = \frac{2\delta\phi}{\sin 2\phi} =$	$\phi =$	$\frac{\delta C}{C} = \frac{2\delta\phi}{\sin 2\phi} =$
0° or 90°	∞	25° or 65°	±0.00131
5 or 85	±0.00589	30 or 60	.00116
10 or 80	.00292	35 or 55	.00107
15 or 75	.00200	40 or 50	.00102
20 or 70	.00156	45 or 45	.00100

Combining all the considerations, then, we see that for the very best work $\phi = 45°$ is preferable; that any deflection between 30° and 45° is, however, nearly as good as 45°, and that any deflection between 30° and 60° is but slightly inferior to these. For most work then it is indifferent as far as $\delta C/C$ is concerned what deflection we use between the limits of 30° and 60°. For work somewhat inferior in accuracy we may use indifferently any angle between 20° and 70°, but should rarely go outside those limits.

Example XXXV. — ***Electro-Static Capacity.*** — *Thomson's or Gott's Method.*—Description of method in Physical Laboratory Notes or in Kempe's Handbook of Electrical Testing. The formula for the method is

$$F_x = \frac{R}{R_x} \cdot F. \quad . \quad . \quad . \quad . \quad . \quad . \quad [125]$$

The battery power used is assumed to be sufficient to enable a change equal to the smallest coil in the resistance-box to be perceived, hence $\delta R = \delta R_x$. The charges in the condensers are greatest when $R + R_x$ is as large as possible, namely, the total resistance r in the box.

By formula [52]

$$\left(\frac{\Delta F_x}{F_x}\right)^2 = \left(\frac{\delta R}{R}\right)^2 + \left(\frac{\delta R_x}{R_x}\right)^2 = \left(\frac{1}{R^2} + \frac{1}{R_x^2}\right)\delta^2 R.$$

This will be a minimum when the last parenthesis is so. As R

and R_x are independent we must differentiate with respect to each successively and equate the coefficients

$$\frac{d}{dR}\left(\frac{1}{R^2}+\frac{1}{R_x^2}\right)=-\frac{2}{R^3}, \quad \frac{d}{dR_x}\left(\right)=-\frac{2}{R_x^3}.$$

Hence

$$-\frac{2}{R^3}=-\frac{2}{R_x^3}; \qquad \therefore R=R_x.$$

The best ratio then is $R/R_x = 1$, and therefore $F/F_x = 1$ or $F = F_x$; that is, it is best to use a known condenser of as nearly as possible the value of the unknown.

Example XXXVI.—Magnetometer.—In measuring $M \div H$ by the magnetometer, the deflecting magnet is placed successively at two distances, r_1 and r_2 from the needle, producing deflections ϕ_1 and ϕ_2. And

$$\frac{M}{H}=\tfrac{1}{2}\cdot\frac{r_1^5\tan\phi_1-r_2^5\tan\phi_2}{r_1^2-r_2^2}. \quad . \quad . \quad . \quad [126]$$

Desired, the best ratio of $r_1 : r_2$; i.e., that which will make $\Delta\frac{M}{H}$ a minimum. The measurements are such that δr_1 and δr_2 may be considered negligible, and $\delta \tan\phi_1 = \delta \tan\phi_2$; thus these are the precision conditions. There are no magnitude conditions which bear on this problem. Then

$$\frac{d}{d(\tan\phi_1)}\left(\frac{M}{H}\right)=\frac{r_1^5}{r_1^2-r_2^2}, \quad \text{and} \quad \frac{d}{d(\tan\phi_2)}\left(\frac{M}{H}\right)=\frac{-r_2^5}{r_1^2-r_2^2}.$$

$$\therefore \Delta^2\frac{M}{H}=\frac{r_1^{10}+r_2^{10}}{(r_1^2-r_2^2)^2}\cdot\delta^2(\tan\phi).$$

To make $\Delta\frac{M}{H}$ a minimum, the fraction in the second member must be a minimum; and we wish, therefore, to find

the value of $r_1 : r_2$, which will make it so. Writing, then, $r_1 : r_2 = n$, or $r_1 = nr_2$, and substituting gives

$$\frac{r_1^{10} + r_2^{10}}{(r_1^2 - r_2^2)^2} = \frac{nr_2^{10} + r_2^{10}}{(nr_2^2 - r_2^2)^2} = r_2^6 \cdot \frac{n^{10} + 1}{(n^2 - 1)^2},$$

in which we have to find the value of n to produce a minimum.

Where the two variables are independent, as r_1 and r_2 in this case, the following proposition may be often of service.

Let $u = f(x, y)$ where x and y are independent variables and f is such a function that it may be separated so that

$$u = \rho(x) \cdot \sigma\left(\frac{x}{y}\right). \quad . \quad . \quad . \quad . \quad . \quad [127]$$

Then the value of $x : y$, which makes $\sigma\left(\frac{x}{y}\right)$ a minimum, is the same as will make u a minimum. For x cannot be expressed as a function of $\frac{x}{y}$, and, therefore, $\rho(x)$ does not enter into the determination of $x : y$ for the minimum. The same is, of course, true if the separation be made into

$$u = \rho'(y) \cdot \sigma'\left(\frac{x}{y}\right). \quad . \quad . \quad . \quad . \quad . \quad [128]$$

Then in the problem in hand we have to find the value of n, which will make $\frac{n^{10} + 1}{(n^2 - 1)^2}$ a minimum.

$$\frac{d}{dn}\frac{n^{10} + 1}{(n^2 - 1)^2} = 3n^{10} - 5n^8 - 2 = 0. \quad \therefore n = 1.32 \text{ approx.}$$

Example XXXVII.—Battery Resistance and E. M. F.— In the ordinary method of measuring the resistance B of a battery, the currents c_1 and c_2 produced by the battery through two known external resistances r_1 and r_2 are observed. Let ρ_1 and ρ_2 represent the total resistances, including battery, leads,

galvanometer, and rheostat; and let E denote the battery E. M. F. Then the formula for the method is

$$B = \frac{c_2 r_2 - c_1 r_1}{c_1 - c_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [129]$$

Desired the best ratio of $c_1 : c_2$.

Following the procedure of page 105, the expression for ΔB must first be obtained.

$$\frac{dB}{dc_1} = \frac{-(c_1 - c_2)r_1 - (c_2 r_2 - c_1 r_1)}{(c_1 - c_2)^2} = \frac{c_2(r_1 - r_2)}{(c_1 - c_2)^2} = \frac{c_2(\rho_1 - \rho_2)}{(c_1 - c_2)^2}.$$

But as c is a function of ρ and the constant E the solution may be made either by writing E/c for ρ or E/ρ for c. The latter will be done. Then

$$\frac{dB}{dc_1} = -\frac{\rho_1^2 \rho_2}{E(\rho_2 - \rho_1)}. \quad \text{Similarly,} \quad \frac{dB}{dc_2} = \frac{\rho_1 \rho_2^2}{E(\rho_2 - \rho_1)}.$$

Then

$$\Delta^2 B = \left(\frac{-\rho_1^2 \rho_2}{E(\rho_2 - \rho_1)}\right)^2 \delta^2 c_1 + \left(\frac{\rho_1 \rho_2^2}{E(\rho_2 - \rho_1)}\right)^2 \delta^2 c_2. \quad . \quad [130]$$

This expression will serve for a galvanometer for which δc is a constant, e.g., a reflecting galvanometer, or one whose scale is uniform and proportional to the current, such as a Weston ammeter. For a tangent galvanometer or any instrument for which $\delta c/c$ is a constant, the expression must be modified to change $\delta^2 c$ to $\delta^2 c/c^2$. This may be done by substituting in the second member E/c_1 for ρ_1 and E/c_2 for ρ_2. This gives

$$\Delta^2 B = \left(\frac{-\rho_1 \rho_2}{\rho_2 - \rho_1}\right)^2 \frac{\delta^2 c_1}{c_1^2} + \left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right)^2 \frac{\delta^2 c_2}{c_2^2}. \quad . \quad . \quad [131]$$

1st. For the reflecting or the Weston galvanometer to find the best ratio $x = c_1/c_2$ substitute in [130] the equivalent of $x = c_1/c_2$, viz., $x = \rho_2/\rho_1$. This gives

$$\Delta^2 B = \frac{\rho_1^4(x^2 + x^4)}{E^2(x-1)^2} \cdot \delta^2 c. \quad . \quad . \quad . \quad . \quad [132]$$

Hence for a minimum

$$\frac{d}{dx}\left(\frac{x^2 + x^4}{(x-1)^2}\right) = 0;$$

$$\therefore x^3 - 2x^2 - 1 = 0,$$

for which by approximate solution $x = 2.2$. The best ratio is then $\rho_2/\rho_1 = 2.2$ or $c_1/c_2 = 2.2$ or $c_1 = 2.2\, c_2$.

To find the best external resistances corresponding to this best ratio we may substitute $x = 2.2$ in [132], giving

$$\Delta^2 B = 20\frac{\rho_1^4}{E^2} \cdot \delta^2 c_1 \text{ approx};$$

$$\therefore \Delta B = \frac{\sqrt{20}}{E} \rho_1^2 \delta c_1.$$

Hence, as far as this can show, ρ_1 should be as small as possible. It must, however, be remembered that if ρ is small compared with B, a serious error will enter from polarization. We should therefore use as small an external resistance as is consistent with the polarization occurring in the given battery.

To give a stated precision $\Delta B/B$ the galvanometer must have such sensitiveness that it can measure the current c_1 with the precision

$$\frac{\delta c_1}{c_1} < \frac{1}{\sqrt{20}} \cdot \frac{B}{\rho_1} \cdot \frac{\delta B}{B}.$$

This expression is of course deduced directly from the above.

2d. For the tangent galvanometer or any for which $\delta c/c$ is a constant, to find the best ratio c_1/c_2. If we proceed by the usual method starting with [131], we shall arrive at the contradiction $1 = 0$, showing that there is no such best ratio. But we may deduce the result which we desire as follows. Simplifying [131], we obtain

$$\Delta^2 B = 2\left(\frac{\rho_1\rho_2}{\rho_2 - \rho_1}\right)^2 \frac{\delta^2 c}{c^2};$$

$$\therefore \Delta B = \sqrt{2}\,\frac{\rho_1\rho_2}{\rho_2 - \rho_1}\,\frac{\delta c}{c} = \sqrt{2}\,\frac{1}{\frac{1}{\rho_1} - \frac{1}{\rho_2}}\,\frac{\delta c}{c}.$$

From the latter it will be seen that ΔB diminishes as ρ_2 increases and as ρ_1 diminishes. Hence ρ_2 should be as large and ρ_1 as small as conditions of range of galvanometer and of polarization will permit. We should therefore use by preference deflections of 60° and 30° or of 70° and 20° on the tangent galvanometer.

The necessary precision and sensitiveness of galvanometer would be determined as follows. Using 60° and 30°, $c_1 : c_2 = 3 : 1$ approx. Then

$$\Delta B = \sqrt{2}\,\frac{3\rho_1^2}{3\rho_1 - \rho_1}\,\frac{\delta c}{c};$$

$$\therefore \frac{\delta c}{c} = \frac{1}{2}\,\frac{B}{\rho_1}\,\frac{\Delta B}{B}.$$

Therefore we must use a galvanometer of a precision at least equal to this value of $\delta c/c$, and of such a "factor" that with the smallest value of ρ_1/B admissible on account of polarization, the deflection will be about 60°. The value of ρ_2 must, of course, be such as to make the second deflection about 30°.

If instead of taking two deflections on one galvanometer we take the second deflection or a much more sensitive galvanometer than the first, but equally precise, that is one which

will measure a very much smaller current but with equal precision $\delta c/c$, we may then make ρ_2 many times as great as ρ_1 and thus improve the conditions of working. This is really what we do in using a potential galvanometer and a current galvanometer in combination as in the method described in the Physical Laboratory Notes.

E. M. F.—By similar demonstrations we may show that for measuring the electromotive force of the battery the following points.

1st. For a galvanometer for which δc is a constant the best ratio of c_1/c_2 is sensibly the same as for measuring B; also that, even using this ratio, ΔE increases with ρ_1, so that ρ_1 should be made as small as is consistent with polarization, just as in measuring B.

2d. For a galvanometer where $\delta c/c$ is constant we must make ρ_2/ρ_1 as large as possible, making ρ_1 large enough to avoid polarization. Of course it is clear that a potential galvanometer is preferable to the two-deflection method for measuring E.

SINES, COSINES, TANGENTS.

	NATURAL.			LOGARITHMIC.		
	Sine.	*Cos.*	*Tan.*	*Sine.*	*Cos.*	*Tan.*
°						
0.0	0.0000	1.0000	0.0000	$-\infty$	0.0000	$-\infty$
0.5	0.0087	1.0000	0.0087	7.9408	0.0000	7.9409
1.	0.0175	0.9998	0.0175	8.2419	9.9999	8.2419
1.5	0.0262	0.9997	0.0262	8.4179	9.9999	8.4181
2.	0.0349	0.9994	0.0349	8.5428	9.9997	8.5431
2.5	0.0436	0.9990	0.0437	8.6397	9.9996	8.6401
3.	0.0523	0.9986	0.0524	8.7188	9.9994	8.7194
4.	0.0698	0.9976	0.0699	8.8436	9.9989	8.8446
5.	0.0872	0.9962	0.0875	8.9403	9.9983	8.9420
10.	0.1736	0.9848	0.1763	9.2397	9.9934	9.2463
20.	0.3420	0.9397	0.3640	9.5341	9.9730	9.5611
30.	0.5000	0.8660	0.5774	9.6990	9.9375	9.7614
40.	0.6428	0.7660	0.8391	9.8081	9.8843	9.9238
45.	0.7071	0.7071	1.0000	9.8495	9.8495	0.0000
50.	0.7660	0.6428	1.1918	9.8843	9.8081	0.0762
60.	0.8660	0.5000	1.7321	9.9375	9.6990	0.2386
70.	0.9397	0.3420	2.7475	9.9730	9.5341	0.4389
80.	0.9848	0.1736	5.6713	9.9934	9.2397	0.7537
90.	1.0000	0.0000	∞	0.0000	$-\infty$	∞

CONSTANTS.

1 metre in inches (U. S. C. S., 1892)..... = 39.37 00
1 inch in millimetres..... = 25.40 05
1 kilogramme in pounds avoirdupois (U. S. legal) ... = 2.20 460
1 pound avoirdupois in kilogrammes (U. S. legal).... = 0.45 359 7
e = base of Naperian logarithms..... = 2.71 828 18
$1/e$ = modulus of common logarithms..... = 0.43 429 45
Radius is equal in length to an arc of..... = 57°.29 578
Arc of 1° in terms of radius..... = 0.01 745 329
Watts per horse-power (see p. 132).... ..... = 746.
Small calories per second per watt (see p. 111)..... = 0.23 87

SQUARES, CUBES, RECIPROCALS.

No.	Square.	Cube.	Recip.	No.	Square.	Cube.	Recip.
1.0	1.00	1.00	1.00	5.5	30.3	166.	.182
1.1	1.21	1.33	0.909	5.6	31.4	176.	.179
1.2	1.44	1.73	.833	5.7	32.5	185.	.175
1.3	1.69	2.20	.769	5.8	33.6	195.	.172
1.4	1.96	2.74	.714	5.9	34.8	205.	.169
1.5	2.25	3.38	.667	6.0	36.0	216.	.167
1.6	2.56	4.10	.625	6.1	37.2	227.	.164
1.7	2.89	4.91	.588	6.2	38.4	238.	.161
1.8	3.24	5.83	.556	6.3	39.7	250.	.159
1.9	3.61	6.86	.526	6.4	41.0	262.	.156
2.0	4.00	8.00	.500	6.5	42.3	275.	.154
2.1	4.41	9.26	.476	6.6	43.6	287.	.152
2.2	4.84	10.6	.455	6.7	44.9	301.	.149
2.3	5.29	12.2	.435	6.8	46.2	314.	.147
2.4	5.76	13.8	.417	6.9	47.6	329.	.145
2.5	6.25	15.6	.400	7.0	49.0	343.	.143
2.6	6.76	17.6	.385	7.1	50.4	358.	.141
2.7	7.29	19.7	.370	7.2	51.8	373.	.139
2.8	7.84	22.0	.357	7.3	53.3	389.	.137
2.9	8.41	24.4	.345	7.4	54.8	405.	.135
3.0	9.00	27.0	.333	7.5	56.3	422.	.133
3.1	9.61	29.8	.323	7.6	57.8	439.	.132
3.2	10.2	32.8	.313	7.7	59.3	457.	.130
3.3	10.9	35.9	.303	7.8	60.8	475.	.128
3.4	11.6	39.3	.294	7.9	62.4	493.	.127
3.5	12.3	42.9	.286	8.0	64.0	512.	.125
3.6	13.0	46.7	.278	8.1	65.6	531.	.123
3.7	13.7	50.7	.270	8.2	67.2	551.	.122
3.8	14.4	54.9	.263	8.3	68.9	572.	.120
3.9	15.2	59.3	.256	8.4	70.6	593.	.119
4.0	16.0	64.0	.250	8.5	72.3	614.	.118
4.1	16.8	68.9	.244	8.6	74.0	636.	.116
4.2	17.6	74.1	.238	8.7	75.7	659.	.115
4.3	18.5	79.5	.233	8.8	77.4	681.	.114
4.4	19.4	85.2	.227	8.9	79.2	705.	.112
4.5	20.3	91.1	.222	9.0	81.0	729.	.111
4.6	21.2	97.3	.217	9.1	82.8	754.	.110
4.7	22.1	104.	.213	9.2	84.6	779.	.109
4.8	23.0	111.	.208	9.3	86.5	804.	.108
4.9	24.0	118.	.204	9.4	88.4	831.	.106
5.0	25.0	125.	.200	9.5	90.3	857.	.105
5.1	26.0	133.	.196	9.6	92.2	885.	.104
5.2	27.0	141.	.192	9.7	94.1	913.	.103
5.3	28.1	149.	.189	9.8	96.0	941.	.102
5.4	29.2	157.	.185	9.9	98.0	970.	.101

	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5052	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5515	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

	0	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
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